CALCULATION OF THE AREAS OF POLYGONAL SURFACES IN THE TOPOGRAPHIC PLAN USING COMPLEX NUMBERS

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Abstract: *The correlations between the topographic plan and the complex plan as well as those between topographic and trigonometric circles enable the approach of the problems of areas, distances, angles and various other topographical issues from the point of view of complex numbers-relying geometry. This paper presents an approach of polygonal surface areas situated in the topographic plan through employing both a classical method as well as the complex numbers method.*

Key words: *topographic plan, complex plan, topographic circle, trigonometric circle, areas*

GIS programs designed in order to calculate polygon areas in the topographic plan use Simpson's rule.

Therefore, when considering a polygon $A_1A_2\ldots.A_n$ located in the first quadrant, not necessarily convex, but whose vertices can be traversed in accordance with a clockwise motion (topographic), then its area, which we mark with $S_{A_1A_2\ldots.A_n}$, is calculated with the formula:

$$S_{A_1A_2\ldots.A_n} = \sum_{i=1}^{n-1}(x_i - x_{i+1})\left(\frac{y_i+y_{i+1}}{2}\right) + (x_n - x_1)\left(\frac{y_n+y_1}{2}\right)$$

Referring to a particular case, in the case of the polygon $A_1A_2A_3A_4A_5$ in *Figure 1*, we have a sum of positive or negative right angle trapezoid areas.

$$S_{A_1A_2A_3A_4A_5} = -S_{B_1B_2A_2A_3} - S_{B_2B_3A_3A_2} - S_{B_3B_4A_4A_3} + S_{B_5B_4A_4A_5} + S_{B_1B_5A_5A_1}$$

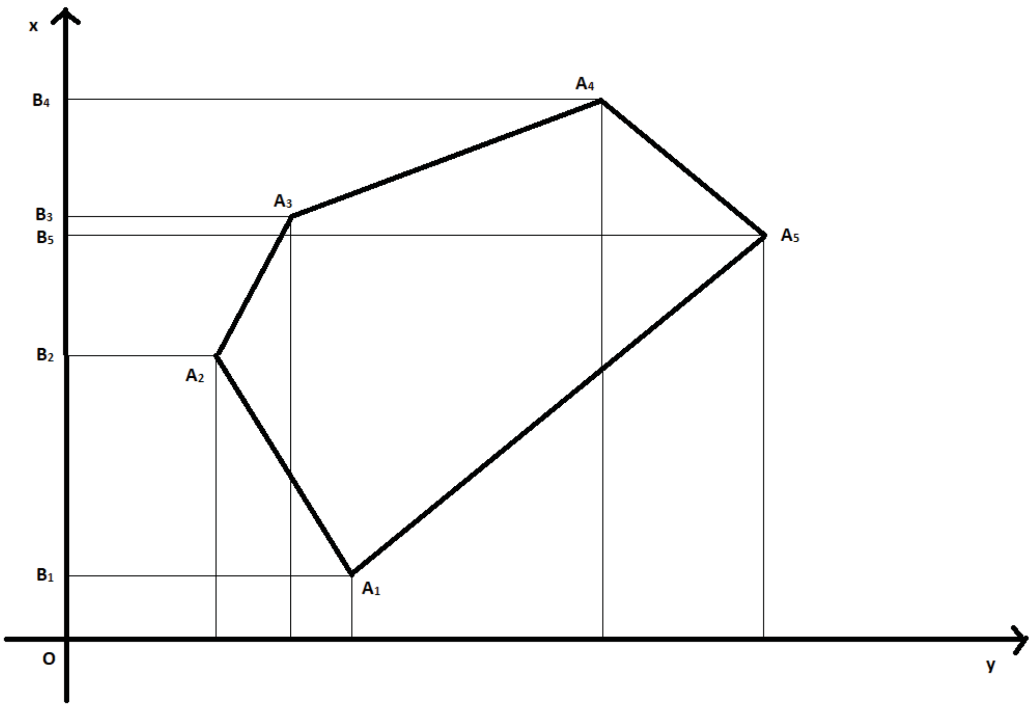

Figure 1

The first problem we encounter relates to the traversing in accordance with a clockwise motion of the polygon's vertices, especially if it is concave, as in *Figure 2*. In order to solve this problem, we can place an observer - O_1 (a topographic device) – inside the polygon so that it can point to the vertices $A_1, A_2, \ldots, A_n$ in this order and in a clockwise motion. In the case of Figure *2,* O_1 must be placed inside the A_1MA_4N polygon.

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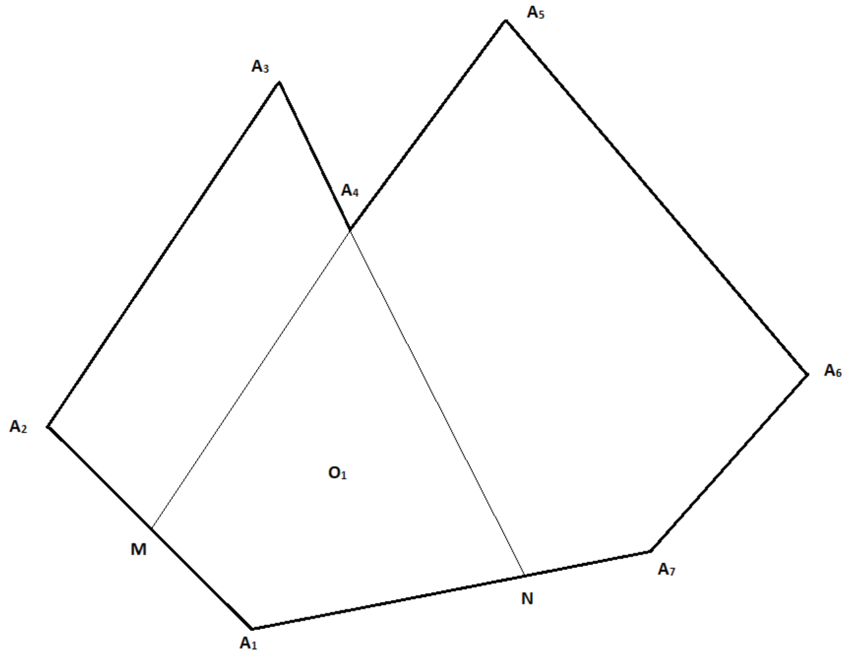

Figure 2

If the placing of O_1 is not possible, we will divide the initial polygon in two or more polygons conveniently joining non-adjacent vertices so that the traversal of the new polygon vertices occurs along the topographic direction. The area of the initial polygon represents the sum of the areas of the new polygons.

In order to approach the areas using complex numbers we must start by emphasizing the bijective correspondence between the set of the complex numbers and the complex plan. In the complex plan, the *Ox* axis is the axis of the real numbers; while the *Oy* axis is that of the pure imaginary numbers (these axes are interchanged for the topographic plan).

Any complex number $z_i = x_i + iy_i$ can be uniquely represented by a point $A_i(x_i, y_i)$ in the complex plan, denoted A_{z_i} and which we will refer to as z_i.

One of the relevant applications of complex numbers in geometry is given by the calculation of the area of a triangle when one of the vertices is placed in the complex plane origin O. (*Figure 3*).

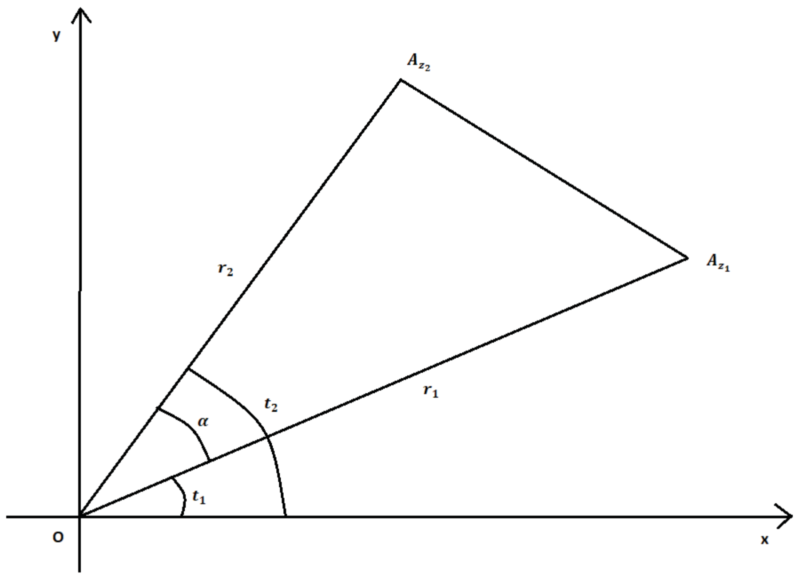

Figure 3

In *Figure 3* A_{z_1} and A_{z_2} are the images of complex numbers z_1 and z_2 expressed in trigonometric form: $z_1 = r_1(\cos t_1 + \mathrm{i}\sin t_1)$ and $z_2 = r_2(\cos t_2 + \mathrm{i}\sin t_2)$ with $r_1 = OA_{z_1}$ and $r_2 = OA_{z_2}$.

Similarly, A_{z_1} and A_{z_2} represent the other two vertices of the triangle of area $S_{OA_{z_1}A_{z_2}}$.

$$S_{OA_{z_1}A_{z_2}} = \frac{r_1 r_2 \sin\propto}{2} \tag{1}$$

Calculating :

$\overline{z_1} z_2 = r_1 r_2 (\cos t_1 - \mathrm{i}\sin t_1)(\cos t_2 + \mathrm{i}\sin t_2) = r_1 r_2 \cos(t_2 - t_1) + i r_1 r_2 \sin(t_2 - t_1)$

$\overline{z_1} z_2 = r_1 r_2 \cos\propto + i\, r_1 r_2 \sin\propto$,

where $\bar{z}$ is the complex conjugate of number *z*.

In conclusion, while using (1) we can write that:

$$S_{OA_{z_1}A_{z_2}} = \frac{Im\,\overline{z_1} z_2}{2} \tag{2}$$

if $A_{z_1}A_{z_2}$ is traversed in trigonometric sense (counter clockwise) and where *Im z* is the coefficient of the imaginary component of the complex number *z*.

In order not to be constrained by the trigonometric sense, we can consider:

$$S_{OA_{z_1}A_{z_2}} = \frac{|Im\,\overline{z_1} z_2|}{2}.$$

Generalizing for a polygon $A_{z_1}A_{z_2} \dots A_{z_n}$ whose vertices are traversed in trigonometric sense and for which the origin O is located in its interior, taking into account its division into triangles with the vertex in the plane origin O and the formula (2) for area calculation we will have the formula:

$$S_{A_{z_1}A_{z_2}\dots A_{z_n}} = \frac{Im\left(\sum_{i=1}^{n-1} \overline{z_i} z_{i+1} + \overline{z_n} z_1\right)}{2} \tag{3}$$

$$S_{A_{z_1}A_{z_2}\dots A_{z_n}} = \left|\frac{Im\left(\sum_{i=1}^{n-1} \overline{z_i} z_{i+1} + \overline{z_n} z_1\right)}{2}\right| \tag{4}$$

if the traversing is made according to a clockwise sense (trigonometrically inversed) or in the cases when it is made in the same sense, with said direction not being specified. The previous formula was determined by the Bulgarian mathematician Kiril Docev for convex polygons represented in the complex plane with the origin *O* inside the polygon. Taking into account the ability of dividing into disjoint triangles with a vertex in *O*, the convexity is not mandatory.

This formula can also be used in the case of the topographic plan. We start from the observation that, in the case one draws the topographic plan and the topographic circle on a transparent piece of paper, an observer - situated behind the paper, with his feet parallel to the *Ox* axis and looking towards the direction of the *Oy* axis - observes the complex plan as well as the trigonometric circle (we can disregard the fact that the measuring unit for the trigonometric circle is the sexagesimal degree, while the topographic circle uses centesimal degrees).

Thus, we can create a bijective association between the points $A_i(x_i, y_i)$ with $i = \overline{1,n}$ that represents the polygon's vertices in the topographic plan and the points A_{z_i}, $i = \overline{1,n}$ from the complex plans, points that at the same time represent the images of complex numbers $z_i = x_i + i y_i$ cu $i = \overline{1,n}$. Concomitantly, the trigonometric sense (counter clockwise) from the complex plan is the same as the topographic sense from the topographic plan (clockwise).

In conclusion, if the polygon $A_1 A_2 \dots . A_n$ includes the point *O*, the origin of the topographic plan, and the vertices are traversed in accordance with the topographic sense, we are able to write that:

$$S_{A_1 A_2 \dots . A_n} = S_{A_{z_1}A_{z_2}\dots A_{z_n}} = \frac{Im\left(\sum_{i=1}^{n-1} \overline{z_i} z_{i+1} + \overline{z_n} z_1\right)}{2}.$$

If the polygon $A_1 A_2 \dots . A_n$ includes the point *O*, the origin of the topographic plan and the vertices are traversed in the same, unspecified, sense we may write that:

$$S_{A_1 A_2 \dots . A_n} = S_{A_{z_1}A_{z_2}\dots A_{z_n}} = \left|\frac{Im\left(\sum_{i=1}^{n-1} \overline{z_i} z_{i+1} + \overline{z_n} z_1\right)}{2}\right|.$$

In the general case, *O* is not included in the polygon $A_1 A_2 \dots . A_n$. We will then proceed as in *Figure 2* in order to determine a point O_1 situated inside the polygon $A_1 A_2 \dots . A_n$ so that the traversing of the vertices can be made in the same sense. Having $O_1(x_{O_1}, y_{O_1})$ we will displace the polygon $A_1 A_2 \dots . A_n$ in the direction, sense and length of vector $\overrightarrow{O_1 O}$, thus obtaining polygon $A_1' A_2' \dots A_n'$ congruent with the initial polygon, having *O* (the origin of the topographic plane) in its interior. The vertices A_i', $i = \overline{1,n}$ of the new polygon will have the coordinates $A_i'(x_i - x_{O_1}, y_i - y_{O_1})$ in the topographic plan and correspond to point $A_{z_i'}$ from the complex plan, points which are the complex images of complex numbers $z_i' = (x_i - x_{O_1}) + i(y_i - y_{O_1})$ created by the displacement using $\overrightarrow{O_1 O}$.

To conclude:

$$S_{A_1 A_2 \dots . A_n} = S_{A_1' A_2' \dots A_n'} = S_{A_{z_1'} A_{z_2'} \dots A_{z_n'}} = \frac{\left|Im\left(\sum_{i=1}^{n-1} \overline{z_i'} z_{i+1}' + \overline{z_n'} z_1'\right)\right|}{2}.$$

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