POSITIONAL SYNTHESIS OF FOUR-BAR MECHANISM

Vasile ZAMFIR, Horia VÎRGOLICI**, Olimpiu STOICUȚA****

Abstract In the paper we present the positional synthesis of the four-bar linkage as part of mining machines and equipment.

Introduction

At the synthesis of four-bar mechanism it is required to determine the relative lengths $a=\dfrac{l_1}{l_0}$, $b=\dfrac{l_2}{l_0}$ and $c=\dfrac{l_3}{l_0}$ and the angles φ_0, ψ_0 so that element (3), considered working element, should generate the function $\psi=F(\varphi)$ on the interval $[\varphi_0, \varphi_n]$, figure 1.

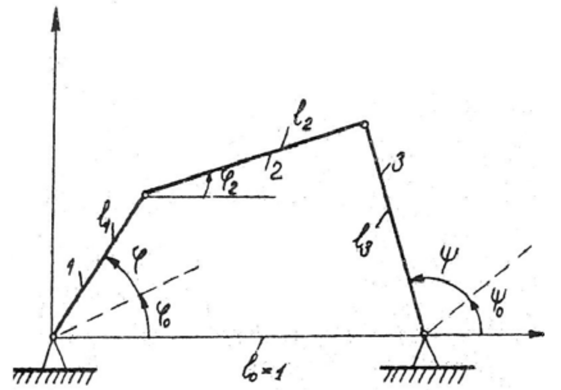

Fig. 1 *The four-bar mechanism and the parameters which defines it*

To this end, five positions of the working element are chosen on the approximation interval $[\varphi_0,\varphi_n]$. They are defined by the abscissae φ_1, φ_2, …, φ_5, to which correspond the positions $\psi_1=F(\varphi_1)$, $\psi_2=F(\varphi_2)$, …, $\psi_5=F(\psi_5)$ of the working element (3).

The function $\psi = f(\varphi; a,b,c,\varphi_0,\psi_0)$ is expressed analytically, obtaining the equation system:

$$\psi_i = f(\varphi_i; a,b,c,\varphi_0,\psi_0);\ i=1,2,...,5 \tag{1}$$

from which the five dimensional and positional parameters of the mechanism a, b, c, φ_0, ψ_0 can be obtained.

From calculations, the system of implicit equations (2) is preferred in place of system (1):

$$f(\varphi_i,\psi_i)=0;\ i=1,2,...,5 \tag{2}$$

The implicit function $f(\varphi,\psi)$ is obtained squaring the mechanism vectorial contour equation:

$$b=a+c+1 \tag{3}$$

obtaining the position function of the four-bar mechanism, of the form:

$$\begin{aligned}&2ac\cos(\psi+\psi_0-\varphi_0-\varphi)+2a\cos(\varphi_0+\varphi)-\\&-2c\cos(\psi+\psi_0)+b^2-a^2-c^2-1=0\end{aligned} \tag{4}$$

Equation (4) gets various appearances, depending on parameter number (three, four or five) chosen to be calculated; it will appear as a generalized polynomial, as it will be shown in what follows.

Three parameter calculation

For the calculation of three parameters of the mechanism structure, equation (4) is written as the following polynomial:

$$p_0+f_0(\varphi)+p_1f_1(\varphi)+p_2f_2(\varphi)=0 \tag{5}$$

The parameters *a, b, c* (parameters φ_0 and ψ_0 are chosen arbitrarily) are to be found. By identification with equation (4) the following expressions are obtained for coefficients p_j and functions f_j, j=0,1,2:

$$\begin{cases} p_0=\dfrac{b^2-a^2-c^2-1}{2c}\\ p_1=\dfrac{a}{c}\\ p_2=a \end{cases} \tag{6}$$

$$\begin{cases} f_0(\varphi)=\cos(\psi_0+\psi)\\ f_1(\varphi)=\cos(\varphi_0+\varphi)\\ f_2(\varphi)=\cos(\psi_0+\psi-\varphi_0-\varphi) \end{cases} \tag{7}$$

The interpolation nodes in the interval $[\varphi_0,\varphi_m]$ are chosen, by abscissae φ_1, φ_2, φ_3, either arbitrarily or as it will be shown further, by Chebyshev spacing for example.

The functions $\psi_i=F(\varphi_i)$, $i=1,2,3$ are calculated in the three positions of the approximation interval, getting the following linear system of three equations with three unknowns p_j, $j=0,1,2$:

$$\begin{cases} p_0+f_0(\varphi_1)+p_1f_1(\varphi_1)+p_2f_2(\varphi_1)=0\\ p_0+f_0(\varphi_2)+p_1f_1(\varphi_2)+p_2f_2(\varphi_2)=0\\ p_0+f_0(\varphi_3)+p_1f_1(\varphi_3)+p_2f_2(\varphi_3)=0 \end{cases} \tag{8}$$

---

* *Prof.eng. Ph.D – University of Petroşani*
***Lect. Ph.D - Univ. „Spiru Haret" Bucureşti*
*** *Asist.eng. Ph.D – University of Petroşani*

from which the unknowns p_0, p_1 and p_2 are found and by means of the relations (6), the unknown dimensional parameters *a, b, c* of the four-bar mechanism are established.

After having found the parameters *a, b, c*, the function (8) is written as the following equation in $sin(\psi_0+\psi_i)$ and $cos(\psi_0+\psi_i)$, after the function $cos(\psi_0+\psi_i-\varphi_0-\varphi_i)$ is developed adequately:

$$A_i \sin(\psi_0+\psi_i)+B_i\cos(\psi_0+\psi_i)+C_i=0 \qquad (9)$$

from which is found:

$$\psi_0+\psi_i=2arctg\left(\frac{A_i\pm\sqrt{A_i^2+B_i^2-C_i^2}}{B_i-C_i}\right) \qquad (10)$$

where:

$$\begin{cases} A_i=-a\sin\varphi_i \\ B_i=-a\cos\varphi_i+1 \\ C_i=\dfrac{a^2-b^2+c^2+1}{2c}-\dfrac{a}{c}\cos\varphi_i;\ i=1,2,...,n \end{cases} \qquad (11)$$

Taking several values, inclusively the values of the nodal abscissae of angle φ_i on the interval (φ_0,φ_m), with relation (10) the value of corresponding angle ψ_i can be determined and then is found the value of deviation $\Delta\psi_i$, $i=0,1,2,...,n$, whose magnitude is compared with the values of allowable deviation $\Delta\psi_{ad}$:

$$\max|\Delta\psi_i|=\max|\psi_i-\psi|\le\Delta\psi_{ad} \qquad (12)$$

The calculation of four parameters a, b, c, ψ_0 (φ_0 is chosen arbitrarily)

In this case, the mechanism position function (4) can be written (after the adequate development of *cos* function) like this:

$$\begin{aligned} &p_0+f_0(\varphi)+p_1f_1(\varphi)+p_2f_2(\varphi)+ \\ &+p_3f_3(\varphi)+p_4f_4(\varphi)=0 \end{aligned} \qquad (13)$$

where:

$$\begin{cases} p_0=\dfrac{b^2-c^2-a^2-1}{2c\cos\psi_0} \\ p_1=\dfrac{a}{c\cos\psi_0} \\ p_2=tg\psi_0 \\ p_3=a \\ p_4=a\ tg\psi_0=p_2p_3 \end{cases} \qquad (14)$$

$$\begin{cases} f_0(\varphi)=-\cos\psi \\ f_1(\varphi)=\cos(\varphi_0+\varphi) \\ f_2(\varphi)=\sin\psi \\ f_3(\varphi)=\cos(\varphi_0+\varphi-\psi) \\ f_4(\varphi)=\sin(\varphi_0+\varphi-\psi) \end{cases} \qquad (15)$$

It can be noticed that coefficient p_4 is a combination of coefficients p_2 and p_3, therefore the system will be non-linear.

The calculation of four parameters a, b, c, φ_0 (ψ_0 is chosen arbitrarily)

In this case, developing the functions in equation (18), we come to the form (13), where p_j and $f_j(\varphi)$ have the appearance:

$$\begin{cases} p_0=\dfrac{b^2-c^2-a^2-1}{2c\cos\varphi_0} \\ p_1=-\dfrac{c}{a\cos\psi_0} \\ p_2=tg\varphi_0 \\ p_3=c \\ p_4=c\ tg\psi_0=p_2p_3 \end{cases} \qquad (16)$$

$$\begin{cases} f_0(\varphi)=-\cos\varphi \\ f_1(\varphi)=\cos(\psi_0+\psi) \\ f_2(\varphi)=\sin\varphi \\ f_3(\varphi)=\cos(\psi_0+\psi-\varphi) \\ f_4(\varphi)=\sin(\psi_0+\psi-\varphi) \end{cases} \qquad (17)$$

As in the previous case, coefficient p_4 is a combination of coefficients p_2 and p_3, therefore the system will be non-linear.

The calculation of five parameters

After developing all *cos* functions in equation (18), the equation is written under the form:

$$\begin{aligned} &p_0+f_0(\varphi)+p_1f_1(\varphi)+p_2f_2(\varphi)+ \\ &+p_3f_3(\varphi)+p_4f_4(\varphi)+p_5f_5(\varphi)=0 \end{aligned} \qquad (18)$$

where:

$$\begin{cases} p_0=\dfrac{b^2-a^2-c^2-1}{2ac\sin(\varphi_0-\psi_0)} \\ p_1=-\dfrac{\sin\varphi_0}{c\sin(\varphi_0-\psi_0)} \\ p_2=\dfrac{\cos\varphi_0}{c\sin(\varphi_0-\psi_0)} \\ p_3=-\dfrac{\cos\psi_0}{a\sin(\varphi_0-\psi_0)} \\ p_4=\dfrac{\sin\psi_0}{a\sin(\varphi_0-\psi_0)} \\ p_5=\dfrac{p_2p_3-p_1p_4}{p_2p_4+p_1p_3} \end{cases} \qquad (19)$$

$$\begin{cases} f_0(\varphi)=\sin(\psi-\varphi) \\ f_1(\varphi)=\sin\varphi \\ f_2(\varphi)=\cos\varphi \\ f_3(\varphi)=\cos\psi \\ f_4(\varphi)=\sin\psi \\ f_5(\varphi)=\cos(\psi-\varphi) \end{cases} \qquad (20)$$

Due to coefficient p_5, the system is non-linear.

After finding coefficients p_j, $j=1,2,...,5$ the same steps are covered in the case of calculation of three parameters.

Supplementary conditions at the positional kinematic synthesis of the four-bar mechanism

The mechanism designed has to ensure the generation of function $\psi=f(\varphi)$ on the interval $[\varphi_0,\varphi_m]$. Besides, the mechanism has to fulfill certain supplementary conditions: the crank existence (the rotability of the driving element 360^0), the maximum lengths of the elements have to surpass a certain value, respectively their minimum lengths have to be below a certain value, the observance of the transmission angle etc.

Crank conditions

In case there is a rotating crank, two extreme conditions of the mechanism appear, called critical points or dead points, figure 2.

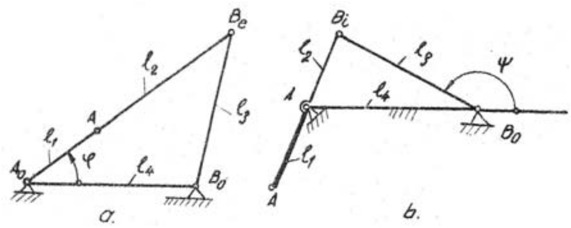

Fig. 2 *Dead points of the four-bar mechanism, where there is a crank*

The following relation is valid for figure 2.a:

$$l_1 + l_2 \le l_3 + l_4 \tag{21}$$

and the following relations for figure 2.b :

$$l_2 - l_1 + l_3 \ge l_4 \tag{22}$$

$$l_3 - (l_2 - l_1) \le l_4 \tag{23}$$

Relation (23) can also be written:

$$l_1 + l_3 \le l_2 + l_4 \tag{24}$$

Supposing crank l_1 is the shortest element. Three cases are possible:

a) If l_4 is the longest element, relation (24) must be fulfilled, because l_4 is the longest element and if relation (23) is fulfilled, even more so is relation (22);

b) If the connecting rod element l_2 is the longest, relation (21) has to be fulfilled, which also includes relation (23), and is no account which of the elements l_3 or l_4 is the longer;

c) If the lever element l_3 is the longest, the inequality (24) has to be fulfilled, and is no account which of the elements l_2 or l_4 is the longest.

From the three cases it can be seen that relation (21) represents the crank condition for all the three cases.

This condition was given by Grasshof in the theorem saying that the necessary and sufficient condition for a four-bar mechanism to admit a crank is that the sum between the length of the shortest and the longest element should be less or at most equal to the length sum of the other two. The four-bar mechanisms that comply with Grasshof theorem are called of complex type. Out of these, by making one of these four links stationary, the same four-bar linkage will yield the following types of four-bar linkages:

- Double crank mechanism, when the shortest element is stationary;
- Double rocker mechanism, if the stationary element is opposite to the shortest element;
- Crank-and-rocker mechanism, when one of the elements adjacent to the shortest one is stationary.

The other mechanisms that do not comply with the conditions of Grasshof theorem are called of simple type, they all being double rocker mechanism, regardless which link is stationary.

The maximum and minimum length of the links

The links relative length has to be between the limits L and $\frac{1}{L}$ (usually L is taken 5).

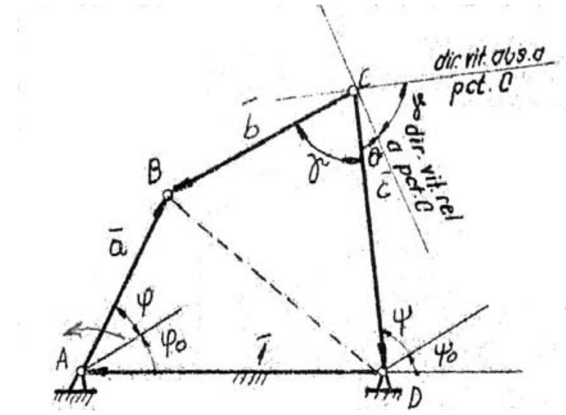

Fig. 3 *The transmission angle for the four-bar linkage*

The following 12 inequalities are given for the four-bar linkage (see figure 3):

$$\begin{cases} a \ge \frac{1}{L}; \quad a \le L \cdot 1; \quad b \ge \frac{1}{L}; \quad c \le L \cdot 1 \\ a \ge \frac{c}{L}; \quad a \le L \cdot c; \quad b \le L \cdot 1; \quad b \ge \frac{c}{L} \\ a \ge \frac{b}{L}; \quad a \le L \cdot b; \quad c \ge \frac{1}{L}; \quad b \le L \cdot c \end{cases} \tag{25}$$

The transmission angle

The transmission angle γ at the four-bar mechanism is made between the absolute velocity direction of point C and the relative velocity direction of point C in relation to point B (figure 3).

Between the transmission angle γ and the pressure angle θ there are the relations (these two angles being complement):

$$\begin{cases} \gamma = 90^0 - \theta, & if\ \gamma < 90^0 \\ \gamma = 90^0 + \theta, & if\ \gamma \geq 90^0 \end{cases} \qquad (26)$$

For the mechanism to have good transmission characteristics between the connecting rod b and the working element c, the transmission angle γ should be comprised between the limits:

$$\gamma_{\min} \geq \gamma_{ad} \quad si \quad \gamma_{\max} \leq 180^0 - \gamma_{ad} \qquad (27)$$

The transmission angle γ can be expressed depending on the mechanism parameters and on the input variable φ and the output one ψ, applying the cosinus theorem in the triangles ABD and BCD:

$$(BD)^2 = b^2 + c^2 - 2bc\cos\gamma = 1 + a^2 - 2a\cos(\varphi_0 + \varphi) \qquad (28)$$

or

$$\cos\gamma = \frac{b^2 + c^2 - a^2 - 1}{2bc} + \frac{a}{bc}\cos(\varphi_0 + \varphi) \qquad (29)$$

If the driving element AB takes up positions AB' and AB'' shown in figure 4, the transmission angle γ has the value $\gamma = \gamma_{\min}$ for $\varphi_0+\varphi=0$ and $\gamma = \gamma_{\max}$ for $\varphi_0+\varphi=180^0$, that is when element AB can be crank.

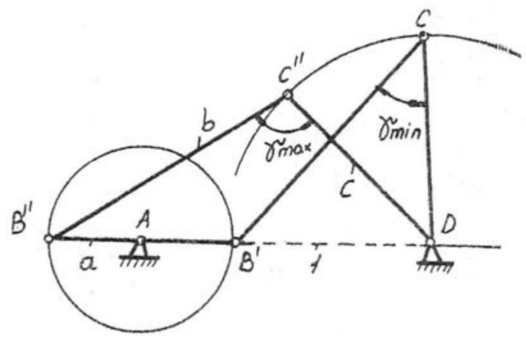

Fig. 4 *The limit transmission angles at the four-bar linkage that allows a crank*

The expressions of the boundary transmission angles can be written with the notations in figure 4:

$$\cos\gamma_{\min} = \frac{b^2 + c^2 - (1-a)^2}{2bc} \qquad (30)$$

$$\cos\gamma_{\max} = \frac{b^2 + c^2 - (1+a)^2}{2bc} \qquad (31)$$

For the transmission angle in a four-bar linkage to be comprised between the boundaries (27) and the driving element AB to be a crank, the following inequalities should be fulfilled:

$$\begin{cases} 2bc\cos\gamma_{ad} \geq b^2 + c^2 - (1-a)^2 \\ 2bc\cos\gamma_{ad} \geq (1+a)^2 - b^2 - c^2 \end{cases} \qquad (32)$$

For the mechanism to be of the crank and rocker type, the crank AB has to be the shortest element. In this case, the last three columns of the inequalities (25) have no longer meaning and therefore, besides the relations (32) referring to the transmission angle, the following conditions have to be fulfilled, referring to the relative minimum and maximum length of the links and the crank rotability:

$$\begin{cases} a \geq \frac{1}{L}; & a < b \\ a \geq \frac{c}{L}; & a < c \\ a \geq \frac{b}{L}; & a < 1 \end{cases} \qquad (33)$$

For the four-bar linkage to have two cranks, as it was seen above, the stationary link should be the shortest element. In this case, besides inequalities (32), the following three inequalities out of the 12 inequalities (25) has to be fulfilled:

$$\begin{cases} 1 < a \leq L \\ 1 < b \leq L \\ 1 < c \leq L \end{cases} \qquad (34)$$

In the case when for a four-bar linkage one or both inequalities (32) are not fulfilled, then the mechanism is of the two rockers type.

In the first case the connecting rod is the shortest element. In this case the following inequalities have to be fulfilled:

$$\begin{cases} a \leq L; & b \geq \frac{1}{L} \\ a \leq Lb; & b \geq \frac{c}{L} \\ a \leq Lc; & c \leq L \end{cases} \qquad (35)$$

In the case when both conditions (32) are not fulfilled, the relation (29) has to used, by means of which the corresponding magnitude of the transmission angle γ is determined for a run of successive values of the input angle φ (corresponding to the oscillation interval of the rocker). In the case when the obtained values for the angle γ are within the prescribed limits, then it is checked if the relations (25) are fulfilled. The above calculations, given with the aim of finding three, four or five parameters, with optimizing conditions regarding the links lengths, the transmission angle and the crank conditions can be easily programmed on the computer.

Numerical example

Let us synthesize the four-bar mechanism (five parameters) for approximation of the interval $\varphi_0 = 0;\ \ \varphi_m = 115^\circ$; in all possible combination cases of samples and multiple interpolation nodes, function:

$$F(\varphi) = \Psi(\varphi) = \frac{1}{250} \cdot \varphi^2 \tag{36}$$

in the next cases: admissibil error $\Delta\psi_{ad}(\varphi) = 20'$; the admissibil transmission angle $\gamma_{ad} = 60^\circ$; the raport between maximal and minimal length of the ellements $L = 5$.

Solution: There are 10 possible combination for the four-bar mechanism:

P-P-P-P-P;PP-P-P-P;P-PP-P-P;PP-PP-P; PP-P-PP; PPP-P-P; P-PPP-P; PP-PPP; PPPP-P; PPPPP

Appling the relation available at subchapter four, we can realise the next diagrams, of the aproximant and aproximated functions, and the aproximate error diagram.

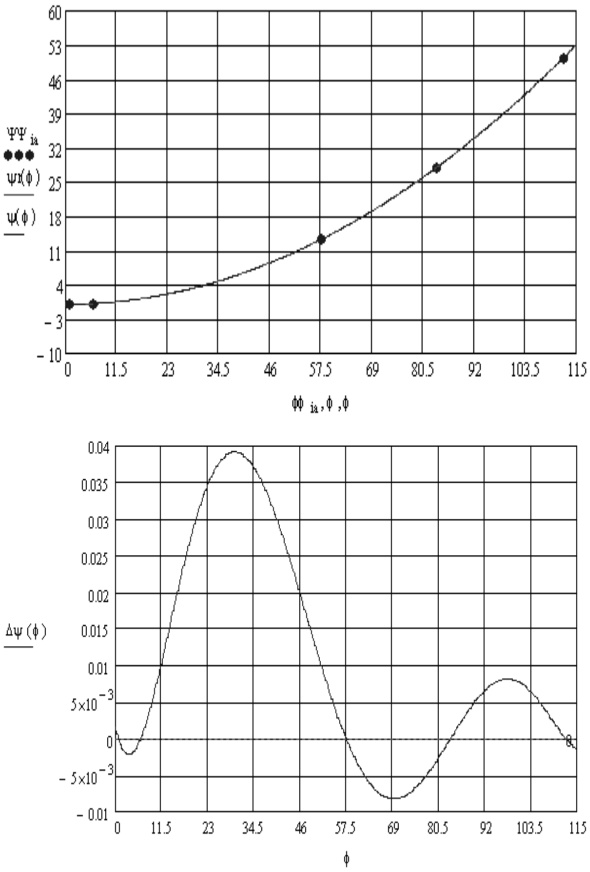

Fig. 5 *P-P-P-P-P case*

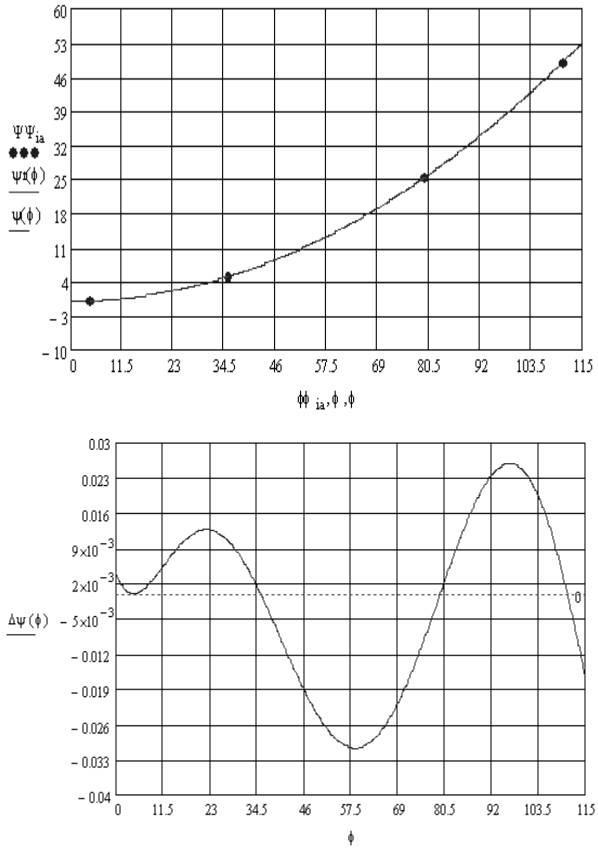

Fig. 6 *PP-P-P-P case*

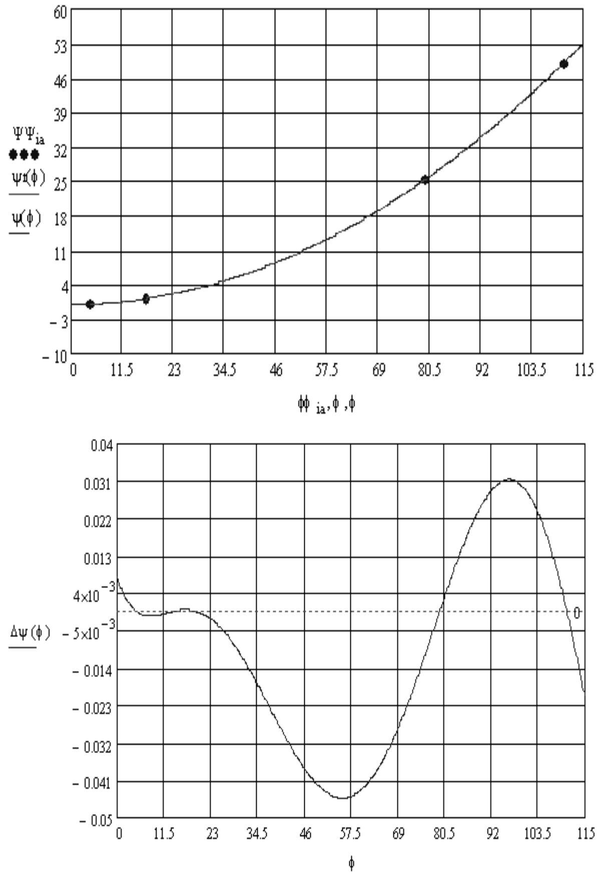

Fig. 7 *P-PP-P-P case*

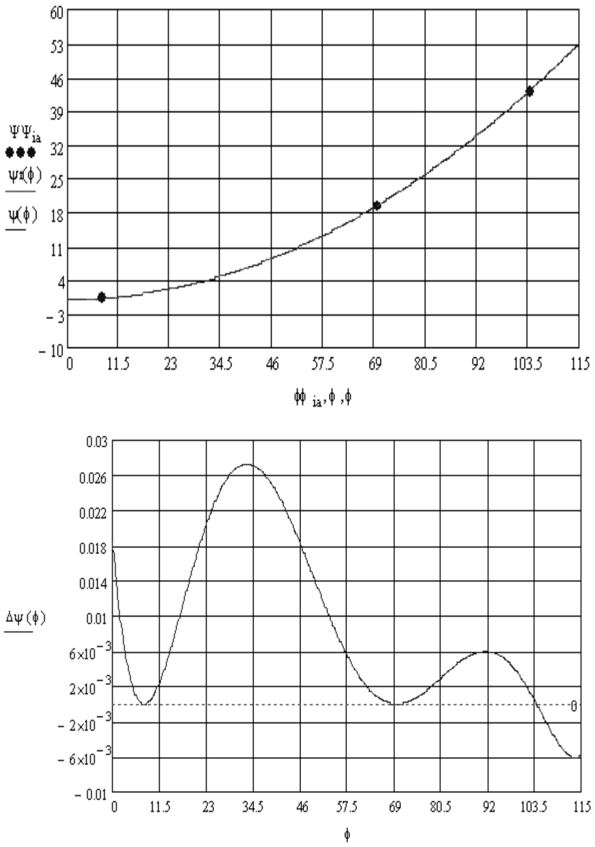

Fig. 8 *PP-PP-P case*

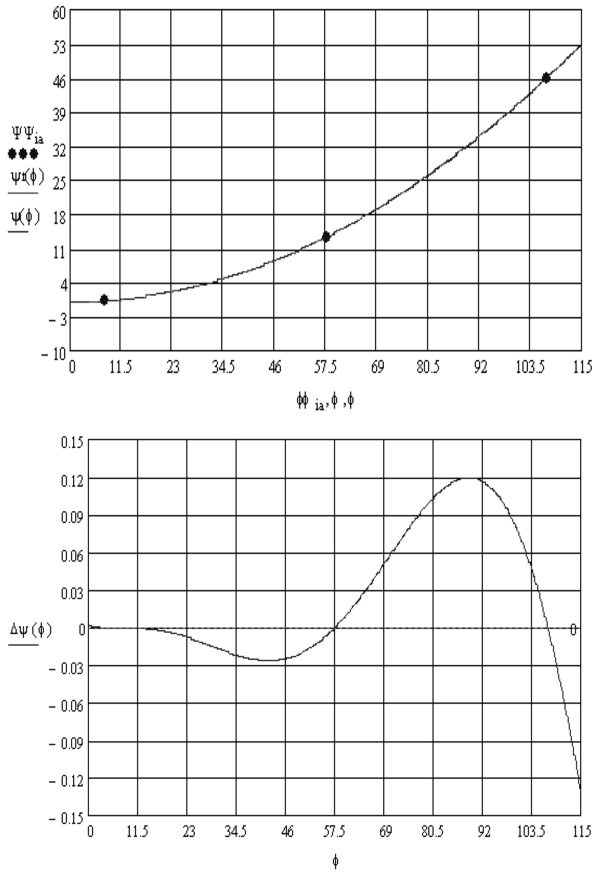

Fig. 10 *PPP-P-P case*

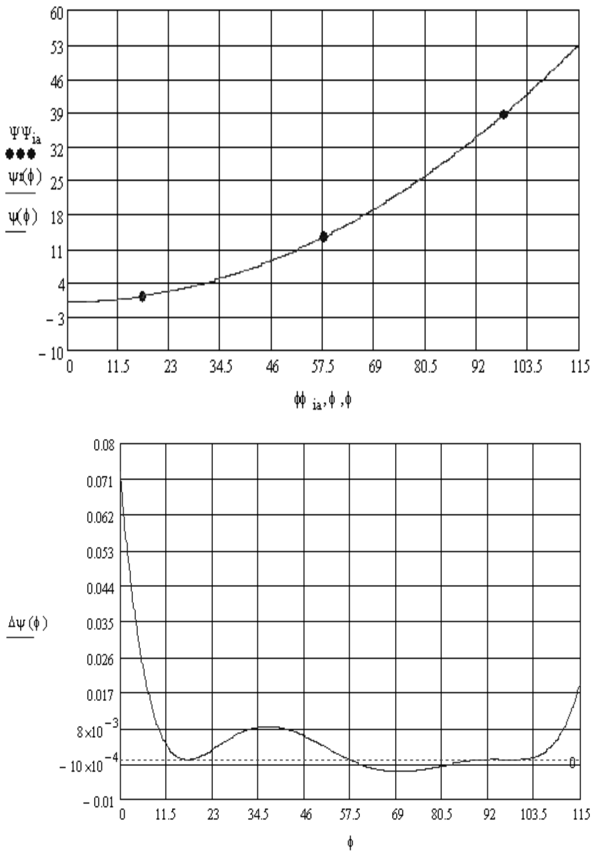

Fig. 9 *PP-P-PP case*

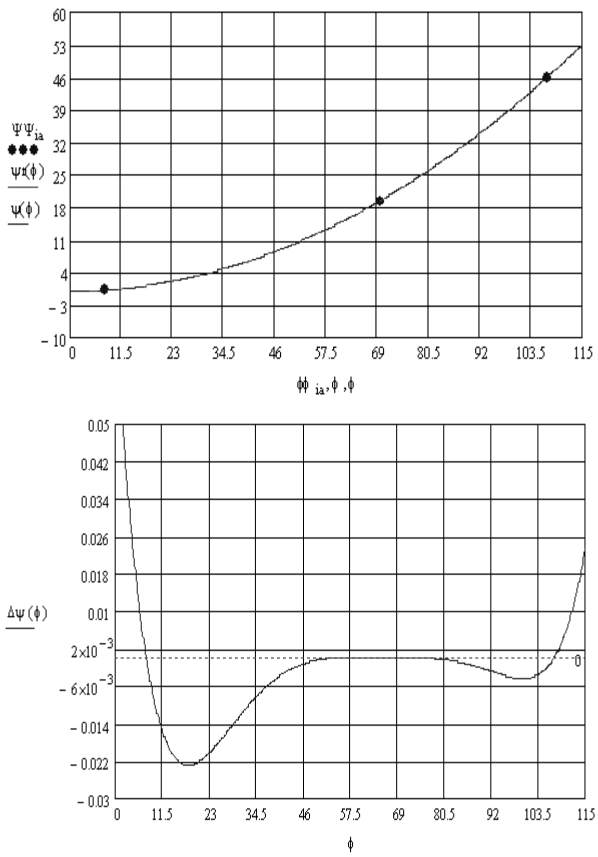

Fig. 11 *P-PPP-P case*

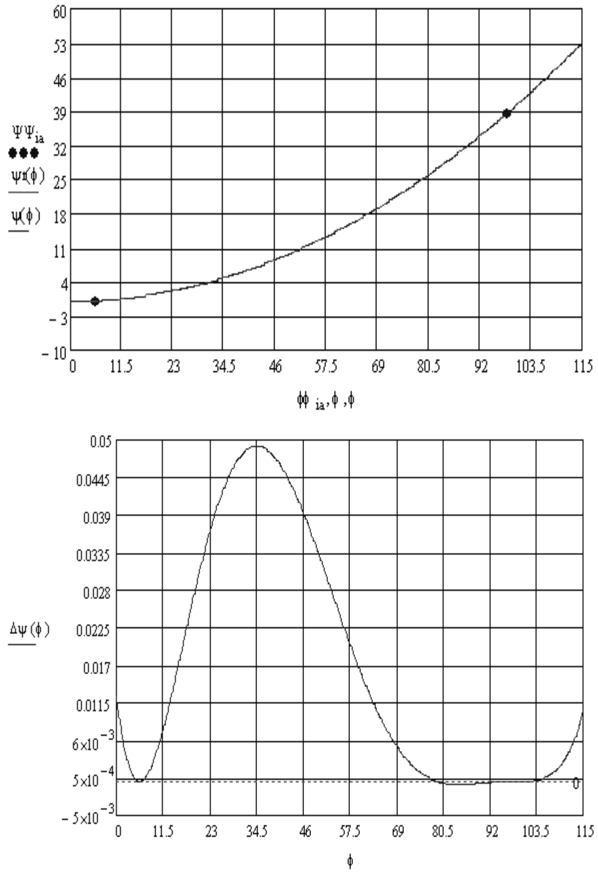

Fig. 12 *PP-PPP case*

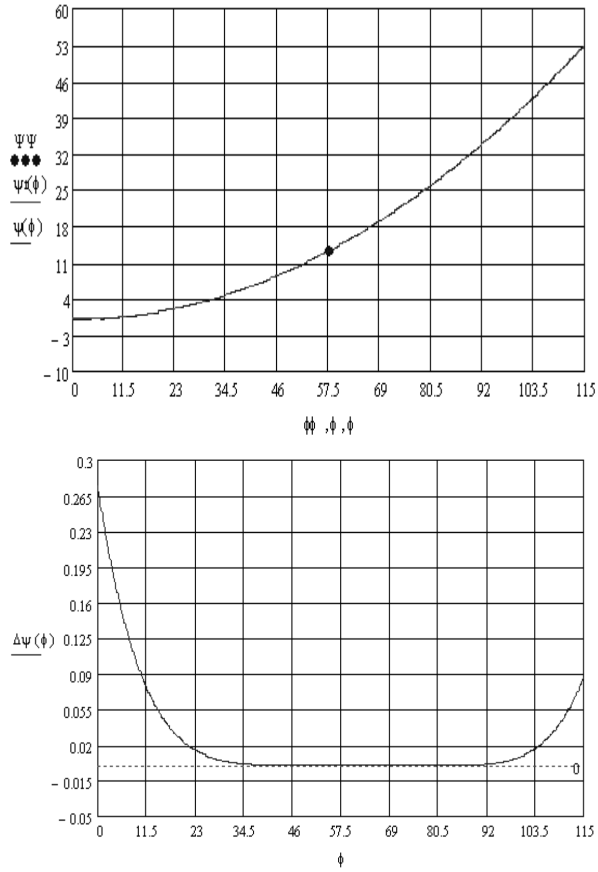

Fig. 14 *PPPPP case*

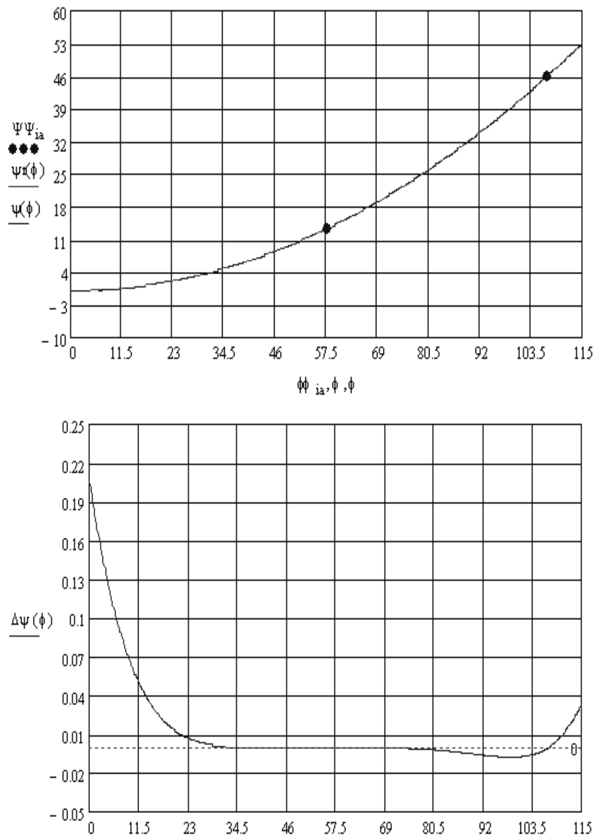

Fig. 13 *PPPP-P case*

Conclusions

The choice of the approximation function depends on circumstances – these being largely determined by the aspect of the function to be approximated, by the desired accuracy, by the preferences and mathematic ability of the person who solves the problem.

The function to be approximated has two aspects:

- the empiric aspect (the function is given by the values table it takes up for certain values of the variable);
- the analytic aspect.

Generally speaking, the functions that are used to approximate the given function can be expressed by *n*-degree polynomials. The polynomial is chosen so that for the same values given to the variable, it should have the same values as the given function. That is to say, the given function curve has (n+1) common points with the representative curve of the *n*-degree polynomial.

However, many times the approximation is done by *p*-degree polynomial, where $p < n$.

In this case, the representative curves will no longer have any common point, but the choice of *p*-degree polynomial is done, setting the condition that the generated diagram should pass as close as possible to the (n+1) points on the given function diagram.

In solving the two aspects of the problem, the number and type of points where the diagrams of the two function touch are of great importance.

If the number of common points is small (small number of parameters) the interpolation method is not satisfactory.

Generally speaking, the interpolation method is used when the approximation is done in several discrete points in the interval. It is recommendable to use the multiple nodes interpolation to diminish the amount of work.

Nowadays, when we can use last generation software, the analytic synthesis has no limits. We can adopt that calculation method that leads most easily to a general equation that can be applied in programming.

Along with the problem of positional kinematic synthesis, a series of supplementary conditions arise; they ensure obtaining the most suitable, simple and operational structural diagram with a low number of elements, so that the designed mechanism should render the demanded motion law as faithfully as possible.

References

1. **Zamfir V., Vîrgolici H.**
Condiții de sinteză prin metoda interpolării
Revista Minelor, vol 17, nr. 1 / 2011

2. **Lazăr M., Pandrea N., Popa D.**
Obtaining Cebâşev-type mechanisms through optimal synthesis based upon some caalculus programs
Mec. Apl., Bulletin of the University of Piteşti, 2001

3. **Georgescu T., Lazăr M.**
Cebâşev-type mechanisms obtained through the optimum synthesis based on some calculus programs
Scientific Bulletin automotiveAutomotive, no. 19, vol 1., University of Piteşti, 2002

4. **Artobolevski I.I., Levitski N.I., Cercudinov S.A.**
Sintez ploskia mehanizmov
Fizmatigiz, Moskva, 1959

5. **Belețki V.**
Rasciot mehanizmov maşin avtomatov piscevâh proizvodstv, „Vişa scola”,
Kiev, 1974

6. **Cercudinov S.A.**
Sintez ploskih şarnirnorîciajnîh mehanizmov
Iz-vo Academii Nauk S.S.S.R., Moskva, 1959

7. **Dancea I.**
Programarea calculatoarelor numerice pentru rezolvarea problemelor cu caracter tehnic şi de cercetare ştiinţifică
Ed. Dacia, Cluj-Napoca, 1973

8. **Hartenberg R.S., Denavit I.**
Kinematic Synthesis of Limkage
McGraw-Hill Series in Mechanical Engineering, New York.

9. **Lazaride Gh., Stere N., Niță C.**
Mecanisme şi organe de maşini
Ed. Didactică şi Pedagogică, Bucureşti, 1970.

10. **Sarkisean Iu.L, Cecean G.S.**
Optimalnîi sintez peredatocinovo cetîrzvenika
Maşino-beledenie, nr.3, 1969.

11. **Tesar D.**
The Generalized Concept of Three Multiply Separated Positions in Coplanara Motion
Journal of Meechanisms, vol.2, 1967, p.461-474

12. **Tesar D.**
The Generalized Concept of Four Multiply Separated Positions in Coplanara Motion
Journal of Meechanisms, vol.3, 1968, p.11-23

13. **Zamfir V.**
Sinteza mecanismelor cu bare articulate plane (Note de curs), fasciculele 1-5
Litografia Institutului de Mine, Petroşani, 1976, 1977.