APPLICATIONS OF LAMI'S THEOREM TO COPLANAR CONCURRENT FORCES IN STATICS

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Abstract: This paper analyses the application of Lami's theorem as an efficient analytical tool for solving statics problems involving coplanar and concurrent force systems. The classical formulation of Lami's theorem for three concurrent forces in equilibrium is presented, together with its mathematical justification and practical limitations. The study is further extended to the case of four concurrent and coplanar forces, where the theorem is applied in combination with the principle of superposition of forces. This approach allows the determination of unknown forces in systems that would otherwise require a larger set of equilibrium equations. Several representative examples from classical mechanics are discussed to illustrate the effectiveness, accuracy, and simplicity of the method, including problems involving friction, inclined planes, and combined loading conditions. The results obtained using Lami's theorem and force superposition are validated through comparison with the classical equilibrium equations of statics, confirming their equivalence. The paper highlights the pedagogical and practical value of Lami's theorem in engineering mechanics.

Keywords: statics; concurrent forces; force equilibrium; superposition principle.

1. INTRODUCTION

Statics is a fundamental branch of classical mechanics that studies the action of forces on bodies at rest and the conditions required to maintain equilibrium. Many engineering and physical systems can be reduced, at least locally, to the analysis of forces acting at a single point, making the study of concurrent force systems particularly important. In this context, analytical methods that simplify equilibrium calculations are highly valuable, both from a theoretical and a practical perspective [1].

Bernard Lami's theorem represents one of the most elegant results in statics, offering a direct relationship between the magnitudes of forces and the angles formed between them

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in systems of concurrent and coplanar forces. Although simple in formulation, the theorem provides a powerful alternative to the classical vector or projection-based equilibrium equations, especially when dealing with geometrically complex force configurations. Named after the French scientist Bernard Lami, the theorem has long been used as an effective tool in engineering mechanics education and problem-solving, due to its clarity, efficiency, and strong geometric interpretation.

The theorem states that if three forces act in one point and are in equilibrium, (see Fig. 1), then each force is proportional with the sinus of the angle between the other two forces.

From a mathematical point of view, it can be thus represented:

$$\frac{F_1}{\sin\alpha}=\frac{F_2}{\sin\beta}=\frac{F_3}{\sin\gamma} \tag{1}$$

Where F_1, F_2 and F_3 are the values of the forces, and α, β and γ are the angles between the forces.

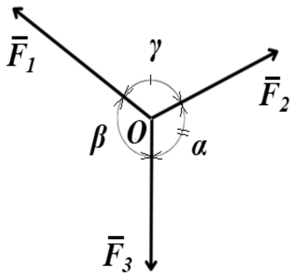

Fig. 1. System of three coplanar and concurrent forces acting at a single point

Figure 1 illustrates a system of three coplanar and concurrent forces acting at a single point. The forces are arranged such that they intersect at the same point and form distinct angles between each pair [2]. The figure represents the equilibrium condition, in which the vector sum of the three forces is zero. This graphical representation provides a clear geometric basis for the application of Lami's theorem, linking the magnitudes of the forces to the angles formed between them.

2. BERNARD LAMI'S THEOREM

We shall further show the demonstration that, if under the action of three concurrent forces a material point (body) stays in equilibrium, each force is directly proportional, in module, with the sinus of the angle formed between the other two forces.

We shall consider three forces, $\overline{P},\overline{Q},\overline{R}$ and acting on a body, *O*. The equilibrium of the body assumes the following condition to be met:

$$\overline{P}+\overline{Q}+\overline{R}=0 \tag{2}$$

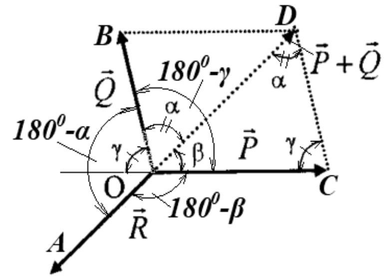

Fig. 2. Point in equilibrium

Namely, the situation in Fig. 2 (the body being *O*) with:

$$\left|\overline{P}+\overline{Q}\right|=\left|\overline{R}\right| \tag{3}$$

We write the theorem of sinuses in triangle *COD* (Fig.2):

$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}=const \quad (=K) \tag{4}$$

Thus:

$$P=K\sin\alpha,\quad Q=K\sin\beta,\quad R=K\sin\gamma \tag{5}$$

with (see Fig. 2):

$$\begin{aligned}\sin\alpha&=\sin\left(180^0-\alpha\right)=\sin\sphericalangle\left(\overline{Q},\overline{R}\right)\\ \sin\beta&=\sin\left(180^0-\beta\right)=\sin\sphericalangle\left(\overline{P},\overline{R}\right)\\ \sin\gamma&=\sin\left(180^0-\gamma\right)=\sin\sphericalangle\left(\overline{P},\overline{Q}\right)\end{aligned} \tag{6}$$

Lami's theorem assumes that the forces are coplanar, concurrent, and the system is in statics equilibrium.

Lami's theorem is applicable specifically to coplanar, concurrent forces systems, with three forces. It is not applicable to non-concurrent forces systems or systems with more or less than three forces.

Lami's theorem provides a simplified method for the analysis of systems of coplanar forces by reducing the number of equations necessary for equilibrium. It is especially useful when we deal with concurrent forces that act on a rigid body[3,4].

We shall further analyze the possibility of applying Lami's theorem for more than three forces. In this sense we present Lami's theorem for four forces found in equilibrium, namely:

If four coplanar, concurrent and non-colinear forces act on a body (Fig. 3), and the body (or the material point) stays in statics equilibrium, then:

$$F_1 \cdot F_4 \sin \alpha' + F_2 \cdot F_3 \sin \gamma' = F_1 \cdot F_2 \sin \beta' + F_3 \cdot F_4 \sin \delta' \quad (7)$$

where F_1, F_2, F_3 and F_4 are the dimensions of the four vectors of α', β', γ' and δ' are the angles between them (see Fig. 3).

Figure 3 illustrates a system of four coplanar and concurrent forces acting at a single point. All forces lie in the same plane and intersect at a common point, satisfying the geometric conditions required for static equilibrium. The angles between consecutive forces are clearly defined, allowing the formulation of a generalized relationship based on Lami's theorem. This configuration highlights the extension of the classical three-force case to systems with four forces, provided equilibrium is maintained. The figure serves as a graphical foundation for deriving the corresponding analytical expression used in the subsequent analysis.

Figure 4 represents the graphical construction obtained by arranging the four concurrent forces head to tail, according to the polygon law of vector addition. The forces form a closed rectangle, which graphically confirms the equilibrium condition of the system, as the resultant force is zero. The interior angles of the rectangle correspond to the angles between the forces shown in Figure 3. This geometric representation allows the equilibrium relationship to be expressed in terms of areas and trigonometric functions[5]. The figure provides a clear visual justification for the generalized form of Lami's theorem applied to four coplanar and concurrent forces.

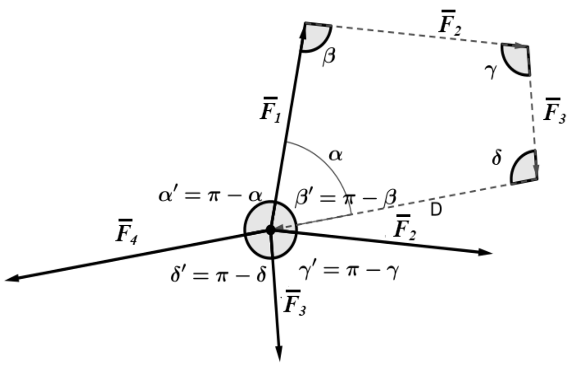

Fig. 3. Concurrence of the four forces

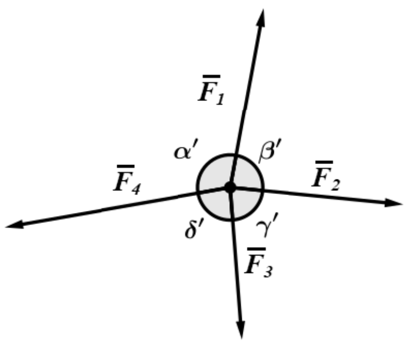

Fig. 4. Rectangle formed by the four

To demonstrate, we consider the rectangle formed by the four vectors, so that the origin of one would correspond to the extremity of the other (graphic law of summing up the vectors, the laws of polygon (Fig. 4) , and we note with A its area. If α', β', γ' and δ' are the inside angles of the rectangle, then its area can be written as the sum of areas of two triangles:

$$A = \frac{1}{2} \cdot F_1 \cdot F_4 \sin \alpha + \frac{1}{2} \cdot F_2 \cdot F_3 \sin \gamma = \frac{1}{2} \cdot F_1 \cdot F_2 \sin \beta + \frac{1}{2} \cdot F_3 \cdot F_4 \sin \delta \quad (8)$$

where $\sin\alpha' = \sin\left(180^0 - \alpha\right) = \sin\alpha$, and similarly for angles β', γ' and δ', equation (8) being identical with equation (7).

Equation (8) is a generalization in the sense that if one of the vectors disappears, the equation that we obtain is that of Lami's theorem. For example, let us assume that $F_3 = 0$, equation (8) is reduced to equation:

$$F_4 \cdot \sin\alpha' = F_2 \cdot \sin\beta' \tag{9}$$

which expresses Lami's theorem.

In statics, for applications, Lami's theorem (for three concurrent forces) can be used in the case of determining one single unknown force, or a force and an angle, both unknown, determining two active forces or two linking forces (reactions). This is possible since with equation (1), two unknowns can be determined.

On the other hand, in the case of using equation (7), equation for four concurrent forces, only one unknown can be determined.

In the following, we shall resort to the principle of superposition (or the principle of overlapping) in physics and systems theory, which express the fact that for any linear system, the answer generated at a certain moment and in a certain position by several stimuli is equal to the sum of the answers generated by each stimulus in part.

A particular case of this principle is the principle of the overlapping of forces in classical mechanic, namely, if several forces act in the same time on a material point(body), each force generates its own acceleration independently of the presence of the other forces, the resultant acceleration being the vectorial sum of the individual accelerations.

The principle of overlapping of forces can be added to the three basic principles of mechanics.

The principle of overlapping forces shows that forces and accelerations are vectorial physical values, which are made up according to the rule of parallelogram.

In the case of statics, since the forces are in equilibrium, the resulting acceleration in null. The forces are made up vectorially, and as per the rule of triangle or polygon, in equilibrium they close, the resultant being null.

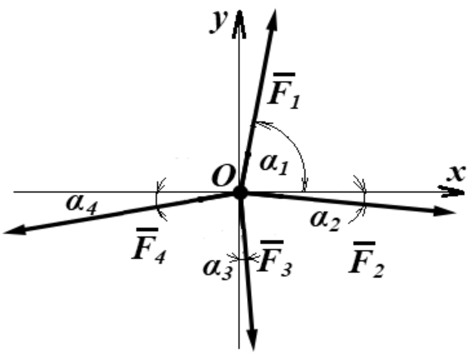

Fig. 5. System of forces considered

Based on the above mentioned principle, considering a system of four concurrent and coplanar forces, of which two are presumed to be known, the angles that the four forces make up with the coordinate axes in the plane and two unknowns as well, we focus on determining the unknowns by successively applying Lami's theorem.

Thus, we consider the system of four forces in Fig. 5, with $\overline{F}_3$, $\overline{F}_4$, α_1, α_2, α_3 and α_4 as being known, $\overline{F}_1$, $\overline{F}_2$ as being unknown.

In order to get to a system of three concurrent forces, in the first case we give up force $\overline{F}_4$ (Fig. 6, a), and in the second case we give up force $\overline{F}_3$ (Fig. 6, b).

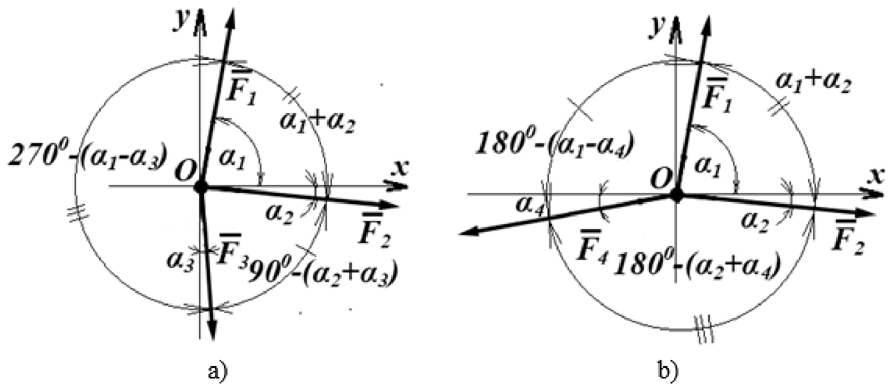

Fig. 6. Systems made up of only three forces for each case

We further use Lami's theorem for each case in part from Fig. 6, and we shall write the equation for the respective case.

Thus, for the first case, according to Fig. 6, we have the equation:

$$\frac{F_1}{\sin\left[90^0-(\alpha_2+\alpha_3)\right]}=\frac{F_3}{\sin(\alpha_1+\alpha_2)}=\frac{F_2}{\sin\left[270^0-(\alpha_1-\alpha_3)\right]}\Leftrightarrow$$
$$\Leftrightarrow\quad\frac{F_1}{\cos(\alpha_2+\alpha_3)}=\frac{F_3}{\sin(\alpha_1+\alpha_2)}=\frac{F_2}{-\cos(\alpha_1-\alpha_3)}\tag{10}$$

whence:

$$F_1=\frac{F_3\cos(\alpha_2+\alpha_3)}{\sin(\alpha_1+\alpha_2)};\ F_2=\frac{-F_3\cos(\alpha_1-\alpha_3)}{\sin(\alpha_1+\alpha_2)}\tag{11}$$

For the second case, according to Fig. 6b, we have the equation:

$$\frac{F_1}{\sin\left[180^0-(\alpha_2+\alpha_4)\right]}=\frac{F_4}{\sin(\alpha_1+\alpha_2)}=\frac{F_2}{\sin\left[180^0-(\alpha_1-\alpha_4)\right]}\Leftrightarrow$$
$$\Leftrightarrow\quad\frac{F_1}{\sin(\alpha_2+\alpha_4)}=\frac{F_3}{\sin(\alpha_1+\alpha_2)}=\frac{F_2}{\sin(\alpha_1-\alpha_4)}\tag{12}$$

whence:

$$F_1 = \frac{F_4 \sin(\alpha_2 + \alpha_4)}{\sin(\alpha_1 + \alpha_2)}; \quad F_2 = \frac{F_4 \sin(\alpha_1 - \alpha_4)}{\sin(\alpha_1 + \alpha_2)} \tag{13}$$

In the end, according to the principle of overlapping forces, the final values of forces $\overline{F}_1$ and $\overline{F}_2$ are:

$$\begin{aligned} F_1 &= \frac{1}{\sin(\alpha_1 + \alpha_2)}\left[F_3 \cos(\alpha_2 + \alpha_3) + F_4 \sin(\alpha_2 + \alpha_4)\right] \\ F_2 &= \frac{1}{\sin(\alpha_1 + \alpha_2)}\left[F_4 \sin(\alpha_1 - \alpha_4) - F_3 \cos(\alpha_1 - \alpha_3)\right] \end{aligned} \tag{14}$$

We shall proceed to results verification, obtained using Lami's theorem together with the principle of overlapping forces, with the help of "classic" conditions of equilibrium in statics.

Thus, the system of forces, according to Fig. 5, in order to be in equilibrium, it is necessary:

$$\overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4 = 0 \tag{15}$$

And scalar equations of equilibrium (projection of forces on system's axes) are:

$$\begin{aligned} F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + F_3 \sin\alpha_3 - F_4 \cos\alpha_4 = 0 \\ F_1 \sin\alpha_1 - F_2 \sin\alpha_2 - F_3 \cos\alpha_3 - F_4 \sin\alpha_4 = 0 \end{aligned} \tag{16}$$

In order to determine force $\overline{F}_1$ we have:

$$\begin{aligned} &\left.\begin{aligned} F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + F_3 \sin\alpha_3 - F_4 \cos\alpha_4 = 0 \,\middle|\, \cdot \sin\alpha_2 \\ F_1 \sin\alpha_1 - F_2 \sin\alpha_2 - F_3 \cos\alpha_3 - F_4 \sin\alpha_4 = 0 \,\middle|\, \cdot \cos\alpha_2 \end{aligned}\right\}(+) \Rightarrow \\ &\Rightarrow F_1\left(\cos\alpha_1 \sin\alpha_2 + \sin\alpha_1 \cos\alpha_2\right) + F_3\left(\sin\alpha_2 \sin\alpha_3 - \cos\alpha_2 \cos\alpha_3\right) - \\ &-F_4\left(\cos\alpha_4 \sin\alpha_2 + \sin\alpha_4 \cos\alpha_2\right) = 0 \Rightarrow \\ &\Rightarrow F_1 == \frac{1}{\sin(\alpha_1 + \alpha_2)}\left[F_3 \cos(\alpha_2 + \alpha_3) + F_4 \sin(\alpha_2 + \alpha_4)\right] \end{aligned} \tag{17}$$

To determine force $\overline{F}_2$ we have:

$$\left.\begin{array}{l} F_1\cos\alpha_1+F_2\cos\alpha_2+F_3\sin\alpha_3-F_4\cos\alpha_4=0\big|\cdot\sin\alpha_1 \\ F_1\sin\alpha_1-F_2\sin\alpha_2-F_3\cos\alpha_3-F_4\sin\alpha_4=0\big|\cdot(-\cos\alpha_1) \end{array}\right\}(+)\Rightarrow$$
$$\Rightarrow F_2\left(\cos\alpha_1\sin\alpha_2+\sin\alpha_1\cos\alpha_2\right)+F_3\left(\sin\alpha_3\sin\alpha_1+\cos\alpha_3\cos\alpha_1\right)-$$
$$-F_4\left(\cos\alpha_4\sin\alpha_1-\sin\alpha_4\cos\alpha_1\right)=0\Rightarrow \tag{18}$$
$$\Rightarrow F_2==\frac{1}{\sin\left(\alpha_1+\alpha_2\right)}\left[F_4\sin\left(\alpha_1-\alpha_4\right)-F_3\cos\left(\alpha_1-\alpha_3\right)\right]$$

It is noticed that the value of forces $\overline{F}_1$ and $\overline{F}_2$ determined with the help of "classic" conditions of equilibrium in statics are identical with those obtained by using Lami's theorem together with the principle of overlapping forces, equations (14). Next, we shall present examples in which Lami's theorem has been used.

3. EXAMPLE REGARDING THE APPLICATION OF LAMI'S THEOREM

In the examples of the following problems, in their solving Lami's theorem has been used, and in some of them, the "classic" conditions of equilibrium in statics as well, and for some of the problems Lami's theorem together with the principle of overlapping forces have been used. This has been done to emphasize the efficiency of the method of solving.

Problem 1:

A particle is seated on a rough board, inclined at angle α as to the horizontal. On it , at angle β to line *CD* (the line of the greatest incline) a force $\overline{F}$ acts, parallel with the plane of the board, as in Fig. 7.

The friction coefficient between the body and the board, μ, being known, and knowing that the equilibrium achieved is at a "limit", it is asked to determine the direction in which the particle will start to move.

Solving:

If the body has weight *G*, the normal reaction on the board in *A* is:

$$N = G\cos\alpha \tag{19}$$

In direction *AD Gsinα* acts. Let us assume that the friction force:

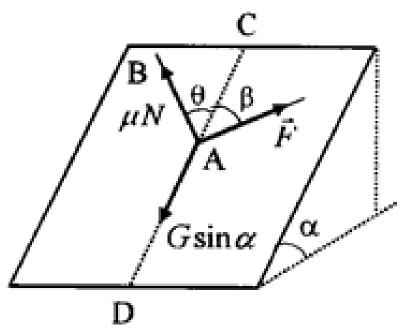

Fig. 7. The position of the particle

$$F_f = \mu N = \mu G\cos\alpha \tag{20}$$

Acts as in Figure 7, in direction *AB*, that is at angle θ as to *DC*. The particle will start to move in direction *BA*.

The three forces being concurrent in A, we can apply Lami's theorem:

$$\frac{\mu N}{\sin\beta}=\frac{G\sin\alpha}{\sin\left(\beta+\theta\right)}=\frac{F}{\sin\theta}, \tag{21}$$

whence:

$$\sin\left(\beta+\theta\right)=\frac{tg\alpha\cdot\sin\beta}{\mu}, \tag{22}$$

equation that determines θ.

4. CONCLUSSION

This paper has presented Lami's theorem as an efficient analytical tool for the study of coplanar and concurrent force systems in statics. The theoretical formulation of the theorem and its geometric interpretation highlight its ability to simplify equilibrium analysis by directly relating force magnitudes to the angles between them. The extension of the theorem to systems with four concurrent forces, achieved through the principle of superposition, demonstrates that Lami's theorem can be effectively integrated into more complex equilibrium analyses without loss of accuracy.

The application illustrated in Example 1 confirms the practical usefulness of the method in solving equilibrium problems involving friction and inclined planes. By reducing the problem to a system of three concurrent forces, Lami's theorem allows a direct and clear determination of the unknown direction of motion, avoiding lengthy vector projections. The results obtained are consistent with the physical behavior of the system, confirming the reliability of the approach. Overall, the study emphasizes both the theoretical relevance and the practical applicability of Lami's theorem in engineering mechanics.

In addition, the geometric nature of Lami's theorem offers a clear physical interpretation of equilibrium conditions, which is particularly useful in educational and engineering contexts. By emphasizing the role of angles between forces rather than their Cartesian components, the method enhances conceptual understanding of force interaction at a point. This perspective is especially relevant for problems where the geometry of the system is well defined, as it allows rapid formulation of equilibrium relations without increasing computational complexity.

Furthermore, the use of Lami's theorem encourages a more intuitive approach to statics problems, complementing classical analytical methods. The approach presented in this paper shows that, when its applicability conditions are satisfied, the theorem can serve as a reliable alternative to traditional equilibrium equations. As illustrated by the analyzed application, Lami's theorem remains a valuable and practical tool for engineers and students alike, supporting both efficient problem-solving and a deeper understanding of force equilibrium in mechanical systems.

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