ASPECTS OF MATRIX SOLUTION OF A VECTOR EQUATION TYPE IN MECHANICS

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Abstract: The paper presents aspects of matrix solution of a type of vector equation in mechanics.

Keywords: sliding vector, vector equation, matrix equation

1. INTRODUCTION

Various situations come up in mechanics, in which vector equations can be formed including a vector product, scalar product or both. Determining the unknown in these equations lies in finding out a vector and, very importantly, studying the relationships between this unknown vector and other vectors. Such vector systems or vector equations can be found in various situations and they can also occur in situations in which certain mechanic processes are influenced by various factors, changing the initial position of a vector or especially changing the position of the vectors, as a result of which, new results of the studied phenomenon can occur. There are several cases in which a vector can be determined having various relationships with other given vectors. Relationships between the unknown vector and other vectors can be expressed by vector equations or systems of vector equations. Vector equations can be of the following types: vector equations representing a vector product and a scalar product; vector equations representing a vector product; vector equation representing a scalar product; vector equations representing three scalar products; combined vector equation.

In presenting the problems of rigid body mechanics, either separate algebraic equation, of the form:

$$\overline{r} \times \overline{v} = \overline{p}\,, \tag{1}$$

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or systems of algebraic vector equations of the form:

$$\overline{r} \times \overline{v} = \overline{p}, \quad \overline{r} \cdot \overline{v} = s, \tag{2}$$

are currently met, in both cases $\overline{r} \cdot$ being an unknown vector, which will be determined by solving equation (1), and system (2), respectively, and vectors $\overline{v}$ and $\overline{p}$, meeting condition $\overline{v} \cdot \overline{p} = 0$, and scalar *s*, having been known.

2. MATRIX SOLUTION OF ALGEBRAIC VECTOR EQUATION OF THE TYPE OF A VECTOR PRODUCT

In the following, we shall deal with matrix solution of an algebraic vector equation of the type given by equation (1).

In the case of determining support *(Δ)* of a sliding vector $\overline{v}$, the starting point is the relationship of definition of the polar moment, where vectors $\overline{v}$ and $\overline{M}_O\left(\overline{v}\right)$ are known, and which will represent a vector equation in the unknown vector $\overline{r}$:

$$\overline{M}_O\left(\overline{v}\right) = \overline{r} \times \overline{v}, \tag{3}$$

Vector equation (3) is multiplied in the left by vector $\overline{v}$:

$$\overline{v} \times \overline{M}_O\left(\overline{v}\right) = \overline{v} \times \left(\overline{r} \times \overline{v}\right), \tag{4}$$

Applying the development formula of the double vector product, we obtain:

$$\overline{v} \times \overline{M}_O\left(\overline{v}\right) = \overline{v}^2 \cdot \overline{r} - \left(\overline{v} \cdot \overline{r}\right) \cdot \overline{v}, \tag{5}$$

whence:

$$\overline{r} = \frac{\overline{v} \times \overline{M}_O\left(\overline{v}\right)}{\overline{v}^2} + \left(\frac{\overline{v} \cdot \overline{r}}{\left|\overline{v}\right|^2}\right)\overline{v},$$

or:

$$\overline{r} = \frac{\overline{v} \times \overline{M}_O\left(\overline{v}\right)}{\overline{v}^2} + \left(\frac{\overline{v} \cdot \overline{r}}{\left|\overline{v}\right|}\right)\overline{u}, \tag{6}$$

Where the $\overline{v}$ vector versor has been noted:

$$\overline{u} = \frac{\overline{v}}{\left|\overline{v}\right|}, \tag{7}$$

It is noticed that the first term of (6) represents the position vector in relation with pole *O* of point $P_0 \in (\Delta)$, $OP_0 \perp (\Delta)$, where *(Δ)* is the support line of $\overline{v}$ vector:

$$\bar{r}_0 = \frac{\bar{v} \times \bar{M}_O(\bar{v})}{\bar{v}^2}, \quad |\bar{r}_0| = \frac{|\bar{M}_O(\bar{v})|}{|\bar{v}|} = OP_0 = b\,, \tag{8}$$

where b is the arm of the vector $b = |\bar{r}| \cdot \sin(\bar{r}, \bar{v})$, and the second term in (6) represents the projection of the position vector $\bar{r}$ on $\bar{v}$ vector support:

$$pr_{\bar{v}}\bar{r} = \frac{\bar{v} \cdot \bar{r}}{|\bar{v}|} = \bar{r} \cdot \bar{u}\,, \tag{9}$$

With this, equation (6) is written in the form of $\bar{r}$ vector equation with an infinity of solutions:

$$\bar{r} = \bar{r}_0 + \bar{u} \cdot pr_{\bar{v}}\bar{r}, \quad \bar{r}_0 \perp \bar{u}\,, \tag{10}$$

having the graphic representation in Fig. 1. In Fig. 1, P_0P represents the position of the $\bar{r}$ vector on the $\bar{v}$ vector support:

$$P_0P = (pr_{\bar{v}}\bar{r}) \cdot \bar{u}\,, \tag{11}$$

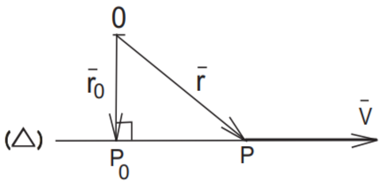

Fig.1. $\bar{v}$ vector support

Noting $\lambda' = pr_{\bar{v}}\bar{r}$, real parameter, equation (11) is written in the form:

$$(\Delta): \bar{r} = \bar{r}_0 + \lambda' \cdot \bar{u}, \quad \lambda' \in \mathbb{R}\,, \tag{12}$$

and represents the vector equation of $\bar{v}$ vector support.

Introducing in (12) $\bar{r}_0$ vector and $\bar{u}$ versor (equations (7), respectively (8)), equation $\bar{v}$ is obtained for the vector support:

$$(\Delta): \bar{r} = \frac{\bar{v} \times \bar{M}_O(\bar{v})}{\bar{v}^2} + \lambda\bar{v}, \quad \lambda \in \mathbb{R}\,, \tag{13}$$

where λ is a real parameter $(\lambda = \lambda' / |\bar{v}|)$.

Substituting in equation (13) the scalar components of $\bar{v}$ and $\bar{M}_O(\bar{v})$, it results:

$$\begin{aligned} x &= \frac{v_y \cdot M_{Oz}(\bar{v}) - v_z \cdot M_{Oy}(\bar{v})}{v_x^2 + v_y^2 + v_z^2} + \lambda \cdot v_x \\ y &= \frac{v_z \cdot M_{Ox}(\bar{v}) - v_x \cdot M_{Oz}(\bar{v})}{v_x^2 + v_y^2 + v_z^2} + \lambda \cdot v_y, \quad \lambda \in \mathbb{R}\,, \\ z &= \frac{v_x \cdot M_{Oy}(\bar{v}) - v_y \cdot M_{Ox}(\bar{v})}{v_x^2 + v_y^2 + v_z^2} + \lambda \cdot v_z \end{aligned} \tag{14}$$

where λ is a real parameter, and $M_{Ox}\left(\overline{v}\right)$, $M_{Oy}\left(\overline{v}\right)$, $M_{Oz}\left(\overline{v}\right)$ are given by equations (scalar components of polar moment):

$$\begin{aligned} M_{Ox}\left(\overline{v}\right) &= yv_z - zv_y \\ M_{Oy}\left(\overline{v}\right) &= zv_x - xv_z \ , \\ M_{Oz}\left(\overline{v}\right) &= xv_y - yv_x \end{aligned} \tag{15}$$

And $\overline{v}$ vector components by equation:

$$\overline{v} = v_x \overline{i} + v_y \overline{j} + v_z \overline{k} \ , \tag{16}$$

To be noted that the polar moment is calculated for the linked vectors as well:

$$\overline{M}_O\left(\overline{v}\right) = \overline{r} \times \overline{v}, \ \ \overline{r} = \overline{OP} , \tag{17}$$

In this case point P being an application of the $\overline{v}$ linked vector.

Considering the way of matrix transcript of the vector product between two vectors, the following matrix equation will correspond to the algebraic vector equation (1):

$$\tilde{\boldsymbol{r}} \cdot \boldsymbol{v} = \boldsymbol{p} , \tag{18}$$

$\tilde{\boldsymbol{r}}$, $\boldsymbol{v}$ and $\boldsymbol{p}$ being matrixes associated to vectors $\overline{r}$, $\overline{v}$ și $\overline{p}$.

The solution of this matrix equation could be obtained by making the following operations:

Equation (18) is multiplied on the left with the asymmetric matrix associated with the known $\boldsymbol{v}$ vector, resulting the following equation as a result of this operation:

$$\tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{r}} \cdot \boldsymbol{v} = \tilde{\boldsymbol{v}} \cdot \boldsymbol{p} , \tag{19}$$

The left member of the previous equation is replaced with the expression resulting from the use of the equation that expresses from a matricial point of view, the associativity of the double vector product in relation to a scalar factor, thus taking the form:

$$\left|\boldsymbol{v}\right|^2 \boldsymbol{r} - \left[\boldsymbol{v}^T \boldsymbol{r}\right] = \tilde{\boldsymbol{v}} \cdot \boldsymbol{p} , \tag{20}$$

The second term from the left member is passed in the right member and the equation is divided by the square of the norm of the $\boldsymbol{v}$ known vector, the equation from below being thus obtained:

$$\boldsymbol{r} = \frac{\tilde{\boldsymbol{v}} \cdot \boldsymbol{p}}{\left|\boldsymbol{v}\right|^2} + \frac{\boldsymbol{v}^T \cdot \boldsymbol{r}}{\left|\boldsymbol{v}\right|^2} , \tag{21}$$

We introduce the notation:

$$\frac{\boldsymbol{v}^{\boldsymbol{T}} \cdot \boldsymbol{r}}{|\boldsymbol{v}|^2} = \lambda \,, \tag{22}$$

λ being an undetermined scalar, as a result of the presence, in the numerator of the fraction from the left member of the previous equation, the $\boldsymbol{r}$ unknown vector. Thus, in this way, the solution of the matrix equation considered results in the following final form:

$$\boldsymbol{r} = \frac{\tilde{\boldsymbol{v}} \cdot \boldsymbol{p}}{|\boldsymbol{v}|^2} + \lambda \boldsymbol{v} \,, \tag{23}$$

Equation (23) is equivalent to equation (13).

In this equation the presence of the λ undetermined factor shows that the system of equations represented by the matrix equation (10) is compatible undetermined.

The solution of the matrix equation (18), under another form, can be obtained starting from the matrix equation:

$$\tilde{\boldsymbol{v}} \cdot \boldsymbol{r} = -\boldsymbol{p} \,, \tag{24}$$

Obtained by reversing the factors from the vectorial product from the left member of the equation (18).

This last equation being written in a developed form, that is:

$$\begin{Vmatrix} 0 r_1 & -v_3 r_2 & v_2 r_3 \\ v_3 r_1 & 0 r_2 & -v_1 r_3 \\ -v_2 r_1 & v_1 r_2 & 0 r_3 \end{Vmatrix} = -\begin{Vmatrix} p_1 \\ p_2 \\ p_3 \end{Vmatrix} , \tag{25}$$

it is noticed, that it is equivalent to a system of three linear nonhomogeneous algebraic equations in the unknowns r_1, r_2 and r_3, with $\tilde{\boldsymbol{v}}$ matrix of singular coefficients, that is, meeting the condition:

$$det[\,\tilde{\boldsymbol{v}}\,] = \begin{vmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 r_1 & v_1 & 0 \end{vmatrix} = 0 \,, \tag{26}$$

It is also noticed that when $\boldsymbol{v} \neq 0$, at least one of the determinants:

$$\begin{vmatrix} 0 & -v_1 \\ v_1 & 0 \end{vmatrix} ; \quad \begin{vmatrix} 0 & v_2 \\ -v_2 & 0 \end{vmatrix} ; \quad \begin{vmatrix} 0 & -v_3 \\ v_3 & 0 \end{vmatrix} ,$$

is not zero, meaning that the rank of the $\tilde{\boldsymbol{v}}$ matrix is $\rho_{\tilde{v}} = 2$.

It results that the system (25) admits two main equations, with main unknowns, represented by two of the coordinates r_1, r_2 and r_3 of $\boldsymbol{v}$ vector.

Forming the enlarged matrix of the system of equation (25), namely matrix:

$$\tilde{\boldsymbol{v}}_p = \begin{Vmatrix} 0 & -v_3 & v_2 & -p_1 \\ v_3 & 0 & -v_1 & -p_2 \\ -v_2 & v_1 & 0 & -p_3 \end{Vmatrix}, \tag{27}$$

it can be easily verified that due to the orthogonality condition of vectors $\boldsymbol{v}$ and $\boldsymbol{p}$, all the characteristic determinants or third order of system (25) of equations are themselves equal to zero. The conclusion can be drawn that the system of equations (25), equivalent to the matrix equation (18) is always compatibly determined.

3. CONCLUSIONS

The paper described the matrix solution of an algebraic vectorial equation, to which a matrix equation corresponds. The paper describes the mathematic part of the matrix solution, for the case of determining the support of a sliding vector.

Substituting the method based on classical vector calculation, in approaching mechanics problems with matrix method, allows the use of modern calculation technique.

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