DECOMPOSITION OF THE ACCELERATION VECTOR IN COMPONENTS THAT ARE NOT OTHOGONALS

RĂZVAN BOGDAN ITU [1], MIREL STOIAN [2], BOGDAN IOAN MARC[3]

Abstract: Kinematic values (kinematic motion parameters) are those values that characterize motion. Many times, the use of another system of coordinate axes, for the study of the point motion, simplify the solving of practical problems; knowing the most usual motion variants is required, which exist in other coordinate systems as well. Thus, the paper presents aspects regarding decomposition of acceleration vector, in plane, in components that are not orthogonal.

Keywords: Motion, decomposition, acceleration

1. INTRODUCTION

Motion is an intrinsic property of matter, in the sense that there is no matter in absolute rest, as well as there is no motion without material support. Modification of the state of motion of a physical state is generally studied as a consequence of the action of the surrounding bodies, or as a result of interactions of parts from the inside of the system. At the beginning, the modification of the state of movement can be studied only descriptively, without taking into consideration the causes that determine it. Such a geometric approach of motion is known as kinematic approach, and the respective chapter in mechanics is called Kinematics.Kinematics of material point studies the mechanical motion of material points, without taking into consideration the masses and forces acting on them. The most important physical values in kinematics are speed and acceleration, defined in relation with various coordinate systems.

2. KINEMATIC PARAMETERS OF MOTION

In order to study the way in which the position of a material point is modified in

[1] *Assoc. Prof. Ph.D. Eng., University of Petroşani,, razvanitu@upet.ro*
[2] *Lecturer Ph.D., University of Petroşani, smstoian@yahoo.com*
[3] *Assistant Ph.D. Eng., University of Petroşani, bogdanmarc94@yahoo.com*

time in relation to another, a system of reference needs to be defined (often called benchmark), considered to be fix in the context of the problem to be studied. The position in relation to the benchmark of the material point the motion of which is in study, is established by the so-called position vector. The latter, most often noted by $r = r(t)$, as function of time, has its origin in the origin of the benchmark, and the peak in the material point in study. Its projections on the axes of the system of reference used, also determine, univocally, the position of a point in space.

The vectorial function $r = r(t)$ represents the law of motion of the material point in vectoral form.

The trajectory is the locus of the successive positions taken by the point in motion. Between the trajectory and the curve on which the point travels there is not always an identity.

The motions can also vary among them by the fact that material points can travel the same distances in different time intervals or different distances in the same time span. These considerations require a new concept, called speed, to be entered. Speed is a vectoral value that establish direction and sense in which motion is done.

Acceleration is a value that shows the speed variation of a point along the motion, as direction, sense and module.

The most often used in the study of point kinematics is the Cartesian system of coordinate axes. This system represents the foundation of building and defining of any other system of coordinates.

3. ORTHOGONAL COORDINATE SYSTEM

In geometry, a system of coordinates is a modality by which to any point i uniquely, an ordered set of real numbers is associated, called the *coordinates* of that point. In the Euclidean space, three coordinates are necessary (abscissa, ordinate and the applicate), in plane, two are necessary (abscissa and ordinate), and to localize the points on line, only one coordinate is required. In analytical geometry, the use of coordinate systems allows transforming the geometry problems in algebra problems.

Orthogonal coordinates are called the ones in which the metric tensor has a diagonal shape.

In mathematics, a tensor is a geometric object that associates in a multi-linear manner the geometric vectors, scalars and other tensors with a resulted tensor. Vectors and scalars that are often used in elementary physics and in engineering applications are considered the simplest tensors. Vectors in dual space of the vectoral space, supplying geometric vectors, are also considered tensors. In this context, the word *geometric* has the aim of underlying the independence of any selection of a system of coordinates. An elementary example of transformation described as tensor is the scalar product, which transforms two vectors into a scalar.

In geometry, and especially in differential geometry, metric tensor is a 2^{nd} order tensor, which makes defining the scalar product of two vectors in each point of a space possible, and which is used to measure lengths and angles. It generalizes Pythagoras' theorem. In a given system of coordinates, the metric tensor can be represented as a symmetric matrix.

In orthogonal systems of coordinates, the surfaces of coordinates are orthogonal the ones with the others. In particular, in Cartesian system of coordinates, the *Ox*, *Oy* and *Oz* axes of coordinates are orthogonal the ones with the others.

The orthogonal coordinates represent a special case of curvilinear coordinates. Most frequently, the orthogonal coordinates are Cartesian coordinates, since in these coordinates, most of the equations have the simplest form. Other systems of orthogonal coordinates are less frequently used, especially for solving problems connected to limit value, such as the problem of thermal conductivity, diffusion, etc. The choice of this or that system of orthogonal coordinates is determined by the system symmetry.

For example, when the problem of propagation of an electromagnetic wave from a point source is solved, it is beneficial to use a system of spherical coordinate system; when the problem of vibration of a membrane is solved, it is preferred a system of cylindric coordinates.

4. PLANE MOTION IN POLAR COORDINATES

Supposing a material point *M* moves in plane *Oxy* (has a plane trajectory) (Fig. 1). Its coordinates can be expressed by the equations:

$$x = r(t)\cos\theta(t), \quad y = r(t)\sin\theta(t), \quad z = 0 \tag{1}$$

where *r(t)* = *OM* (polar radius) and angle $\boldsymbol{\theta}(t)$ (polar angle) made by the polar radius with *Ox* axis of the Cartesian system.

The point movement is defined if coordinates *r* and $\boldsymbol{\theta}$ are known as time functions:

$$r = r(t); \quad \theta = \theta(t), \tag{2}$$

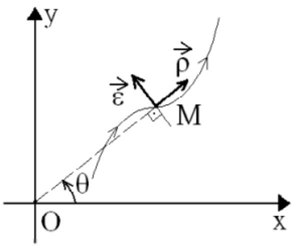

Fig. 1. Polar coordinates

Equations (2) represent parametric equations of the trajectory. By eliminating time between the two equations the analytical expression on the trajectory is obtained (equation of trajectory *f(r,θ)=0*) in polar coordinates: $r = r(\theta)$.

To establish the directions in which speed and acceleration in the system of polar coordinates are projected, we introduce vectors (versors of polar coordinates), $\overline{\rho} = cos\,\theta \cdot \overline{i} + sin\,\theta \cdot \overline{j}$ and $\overline{\varepsilon} = -sin\,\theta \cdot \overline{i} + cos\,\theta \cdot \overline{j}$. Versors $\overline{\rho}$ and $\overline{\varepsilon}$ are orthogonal (see Fig. 1).

We further consider that polar angle θ stays constant and *r* varies. Thus, point *M* describes line *OM* that represents one of the projection directions. In this direction we have chosen the versor $\overline{\rho}$ in the sense of increasing the polar radius.

We then consider that polar radius stays constant. It results that point M describes a circle arc of radius $OM = r$. The second direction tangent to this arc is chosen in point M. In this direction we have chosen versor $\overline{\rho}$ with the positive sense given by the sense of increase of angle θ.

During the motion, versors $\overline{\rho}$ and $\overline{\varepsilon}$ change their direction, thus the axes of the polar coordinate system are mobile, but versors $\overline{\rho}$ and $\overline{\varepsilon}$ stay perpendicular.

For the expression of speed and acceleration in polar coordinates, we have the equations:

$$\overline{v} = \dot{r}\overline{\rho} + r\dot{\theta}\overline{\varepsilon}, \quad \overline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\overline{\rho} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\overline{\varepsilon}, \tag{3}$$

We have:

$$\overline{v} = \frac{d}{dt}\left[r(t)\overline{\rho}(t)\right] = \dot{r}\overline{\rho} + r\dot{\overline{\rho}},$$

but

$$\dot{\overline{\rho}} = \left(\cos\theta \cdot \overline{i} + \sin\theta \cdot \overline{j}\right)' = \dot{\theta}\left[\cos\left(\theta + \frac{\pi}{2}\right)\cdot \overline{i} + \sin\left(\theta + \frac{\pi}{2}\right)\cdot \overline{j}\right] = \dot{\theta}\cdot\overline{\varepsilon},$$

For the second expression in equation (3), by derivation of the first in relation to time t, we get:

$$\overline{a} = \ddot{r}\overline{\rho} + \dot{r}\dot{\overline{\rho}} + \dot{r}\dot{\theta}\overline{\varepsilon} + r\ddot{\theta}\overline{\varepsilon} + r\dot{\theta}\dot{\overline{\varepsilon}} = \ddot{r}\overline{\rho} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\overline{\varepsilon} + r\dot{\theta}\dot{\overline{\varepsilon}},$$

and, since:

$$\dot{\overline{\varepsilon}} = \left(-\sin\theta \cdot \overline{i} + \cos\theta \cdot \overline{j}\right)' = \dot{\theta}\left[\cos(\theta + \pi)\cdot \overline{i} + \sin(\theta + \pi)\cdot \overline{j}\right] = -\dot{\theta}\cdot\overline{\rho},$$

the demonstration is concluded.

5. COMPONENTS OF ACCELERATION IN PLANE MOTION IN NON-ORTHOGONAL COORDINATES

Considering that frequently the use of another system of coordinate axes, for the study of the point motion, simplifies the solving of certain problems, the knowledge of the most usual variants of motion can be studied in other systems of coordinates as well.

In the following, the acceleration vector components in plane motion in coordinates that are not orthogonal (pedal coordinates) are determined. We stay in the frames of the plane motion, and we shall introduce polar coordinates (r,θ). The base is $C = \{\overline{\rho}, \overline{\varepsilon}\}$ where the versor of the vectoral radius $\overline{r}$ is $\overline{\rho}$ and $\overline{\varepsilon} = \overline{k} \times \overline{\rho}$. See Fig. 2.

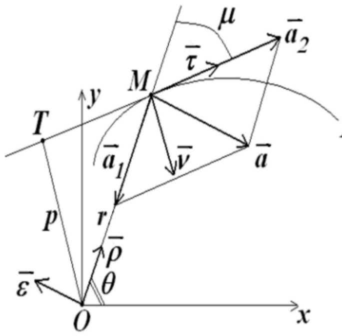

Fig. 2. Polar and pedal coordinates

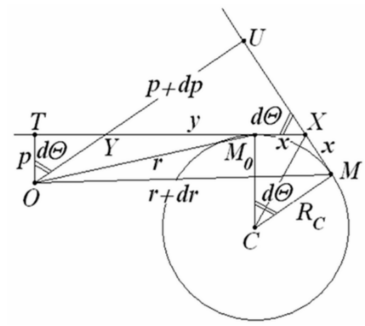

Fig. 3. Curve radius

Thus the following equations take place:

$$\overline{r} = r\overline{\rho},\quad \overline{v} = \dot{r}\overline{\rho} + r\dot{\theta}\overline{\varepsilon},\quad \overline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\overline{\rho} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\overline{\varepsilon} \tag{4}$$

We note with p the distance from origin O, of the system of coordinates to tangent $\Delta\left(M, \overline{r}\right)$ in point M, curve Γ. The pair (r, p) assigns *pedal coordinates* of curve Γ. See Fig.2.

The curve equation in pedal coordinates is given by the expression:

$$k = \frac{1}{r}\left|\frac{dp}{dr}\right| \tag{5}$$

We shall further justify the equation (5), then make the demonstration itself. See Fig. 3.

Thus, knowing that we have values shown in Fig. 3 $\left|\overline{OT}\right| = p$, $\left|\overline{OU}\right| = p + dp$, $\left|\overline{OM_0}\right| = r$, $\left|\overline{OM}\right| = r + dr$ and also angle $\sphericalangle TOY = d\theta$, the equation of the tangent circle radius in points M_0 and M belonging to lines TM_0 and UM is:

$$R_C = \frac{\sqrt{\left(r + dr\right)^2 - \left(p + dp\right)^2} - \sqrt{r^2 - p^2}\cos\theta + p\sin\theta}{\sin d\theta} \tag{6}$$

In the case of our discussion, this is the curve radius in position M_0 of trajectory Γ. To establish equation (6), we enter the notations $x = \left|\overline{MX}\right| = \left|\overline{XM_0}\right|$, $y = \left|\overline{YM_0}\right|$ and it is noticed that $\sphericalangle TOY \equiv \sphericalangle UXY \equiv \sphericalangle MCM_0$.

Then:

$$\begin{aligned}\left|\overline{UM}\right| &= \left|\overline{UX}\right| + \left|\overline{XM}\right| = \left|\overline{YX}\right|\cos\left(\sphericalangle UXY\right) + x = \\ &= \left(x + y\right)\cos d\theta + x = \sqrt{\left|\overline{OM}\right|^2 - \left|\overline{OU}\right|^2}\end{aligned}, \tag{7}$$

$$\left|\overline{UM}\right| = \sqrt{\left(r+dr\right)^2 - \left(p+dp\right)^2}\ , \tag{8}$$

We further have :

$$\left|\overline{TM_0}\right| = \left|\overline{TY}\right| + \left|\overline{YM_0}\right| = \left|\overline{OT}\right| \tan\left(\sphericalangle TOY\right) + y\ , \tag{9}$$

$$\left|\overline{TM_0}\right| = p \tan d\theta + y\ , \tag{10}$$

$$\left|\overline{TM_0}\right| = \sqrt{\left|\overline{OM_0}\right|^2 - \left|\overline{OT}\right|^2} = \sqrt{r^2 - p^2}\ , \tag{11}$$

From equations (10) - (11) and (7) - (8) we get:

$$\begin{cases} y = \sqrt{r^2 - p^2} - p \tan d\theta \\ x = \dfrac{\sqrt{\left(r+dr\right)^2 - \left(p+dp\right)^2} - \sqrt{r^2 - p^2}\cos\theta + p\sin\theta}{2\cos^2 \dfrac{d\theta}{2}} \end{cases},$$

The relationships

$$R_C = \left|\overline{CM}\right| = \frac{\left|\overline{MX}\right|}{\tan\left(\sphericalangle MCX\right)} = \frac{x}{\tan\dfrac{d\theta}{2}}\ , \tag{11}$$

together with x formula, lead to equation (6).

We shall further consider that $\left(dr\right)^2 = \left(dp\right)^2 = dr \cdot dp = 0$, $dp>0$, respectively $\frac{x}{y} = \frac{dp}{y^2} = 0$, $r^2 - p^2 > 1$ and $cos d\theta = 1$, $sin d\theta = d\theta$. this allows us to make the following approximations:

$$\begin{cases} \left|\overline{OY}\right| = \dfrac{\left|\overline{OT}\right|}{\cos\left(\sphericalangle TOY\right)} = \dfrac{p}{\cos d\theta} = p \\ \left|\overline{UY}\right| = \left|\overline{OU}\right| - \left|\overline{OY}\right| = p + dp - \dfrac{p}{\cos d\theta} = dp \end{cases},$$

and

$$\sin d\theta = \sin\left(\sphericalangle UXY\right) = \frac{\left|\overline{UY}\right|}{\left|\overline{YX}\right|} = \frac{dp}{x+y}\ , \tag{12}$$

respectively – as per (9), (11) we have:

$$\sqrt{r^2 - p^2} = \left|\overline{TM_0}\right| = \left|\overline{OT}\right| \tan\left(\sphericalangle TOY\right) + y = \\ = p \tan d\theta + y = p \sin d\theta + y = y + \frac{pdp}{x+y} , \tag{13}$$

But, for any numbers *α, β > 0* so that *α > 1*, $\beta^2 = 0$, we have the approximation:

$$\sqrt{\alpha + \beta} = \sqrt{\alpha} + \frac{\beta}{2\sqrt{\alpha}} ,$$

Taking $\alpha = r^2 - p^2$, $\beta = 2\left(rdr - pdp\right)$, we deduce that:

$$\sqrt{\left(r + dr\right)^2 - \left(p + dp\right)^2} - = \sqrt{r^2 - p^2} \cos d\theta = \\ = \sqrt{r^2 - p^2 + 2\left(rdr - pdp\right)} - \sqrt{r^2 - p^2} = \frac{rdr - pdp}{\sqrt{r^2 - p^2}} ,$$

Then, taking into account (13), we get:

$$\left(x + y\right)\sqrt{\left(r + dr\right)^2 - \left(p + dp\right)^2} - \sqrt{r^2 - p^2} \cos d\theta = \\ = \left(x + y\right)\frac{rdr - pdp}{y + \dfrac{pdp}{x+y}} = \left(rdr - pdp\right)\frac{x + y}{y + \dfrac{pdp}{x+y}} = , \tag{14} \\ = \left(rdr - pdp\right)\frac{\dfrac{x}{y} + 1}{1 + \dfrac{p}{\dfrac{x}{y} + 1} \cdot \dfrac{dp}{y^2}} = rdr - pdp$$

In the end, entering estimations (12), (14) in equation (6), we get:

$$R_C = \frac{rdr}{dp} ,$$

which justifies the "verification" of equation (5). To demonstrate equation (5), we need an intermediary equation regarding value *p*, namely $\dfrac{r^2\left|\dot{\theta}\right|}{v}$. To this end, we start from:

$$\overline{r} \times \overline{\tau} = \left(\overline{OT} + \overline{TM}\right) \times \overline{\tau} = \overline{OT} \times \overline{\tau}, \quad \left(\overline{TM} \times \overline{\tau} = 0, \quad \overline{TM} = \lambda\overline{\tau}\right),$$

and get:

$$p = \left|\overline{OT}\right| = \left|\overline{OT} \times \overline{\tau}\right| = \left|\overline{r} \times \overline{\tau}\right| ,$$

Considering equations (4), we have (with the observation that $\overline{\rho} \times \overline{\varepsilon} = \overline{k}$):

$$\overline{r} \times \overline{\tau} = \pm \frac{\overline{r} \times \overline{v}}{v} = \pm \frac{r^2 \dot{\theta}}{v} \cdot \overline{k} ,$$

we get:

$$p = \frac{r^2 \left|\dot{\theta}\right|}{v} , \tag{15}$$

Further on, equalities take place— cf. [17, p. 72, Theorem 4.3], [16, p. 30] —

$$v^3 k = \left|\overline{v} \times \overline{a}\right| = \left|2\dot{r}^2 \dot{\theta} + r\dot{r}\ddot{\theta} - r\ddot{r}\dot{\theta} + r^2 \dot{\theta}^3\right| , \tag{16}$$

respectively, based on equation (15) we have:

$$\frac{v^3}{r}\left|\frac{dp}{dr}\right| = \frac{v^3}{r}\left|\frac{\dot{p}}{\dot{r}}\right| = \left|\frac{v^2 \dfrac{d}{dt}\left(r^2\dot{\theta}\right) - r^2\dot{\theta}\left(v \cdot \dot{v}\right)}{r \cdot \dot{r}}\right| , \tag{17}$$

entering relationships:

$$\begin{cases} v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \\ v \cdot \dot{v} = \dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta} \end{cases} ,$$

in the third term of the equalities (17), we get:

$$\frac{v^3}{r}\left|\frac{dp}{dr}\right| = \left|2\dot{r}^2 \dot{\theta} + r\dot{r}\ddot{\theta} - r\ddot{r}\dot{\theta} + r^2 \dot{\theta}^3\right| ,$$

which concludes the demonstration of equation (5).

We come back to Fig. 3. In [6, p. 215, probl. 1972] the expression is discussed in polar coordinates of the value of angle μ made up of line *OM* with tangent to trajectory in current point - namely - M -. Thus, starting from:

$$\cos\mu = \overline{\rho} \cdot \overline{\tau} = \overline{\rho} \cdot \frac{\overline{v}}{v} = \frac{\overline{\rho} \cdot \overline{v}}{v} = \frac{\dot{r}}{v} ,$$

respectively, taking into account (15):

$$\sin\mu = \pm\sqrt{1-\cos^2\mu} = \pm\frac{r\dot{\theta}}{v} = \pm\frac{p}{r},$$

we get:

$$\tan\mu = \pm\frac{r\dot{\theta}}{\dot{r}} = \pm r\frac{d\theta}{dr},$$

In particular, function $r = r(\theta)$ satisfies linear differential equation:

$$r' = \pm\cot an\mu \cdot r, \quad unde \ {}' = \frac{d}{d\theta},$$

Let $S = pv = |\overline{r}\times\overline{v}|$. The following result takes place, known as *F. Siacci* theorem, cf. [11, p. 472]: acceleration $\overline{a}$ of particle M is decomposed as per directions $\overline{\rho}$ and $\overline{\tau}$ in accelerations $\overline{a}_1$ and $\overline{a}_2$ so that – see Fig. 2 -:

$$a_1 = |\overline{a}_1| = \frac{S^2 r}{R_C p}, \quad a_2 = |\overline{a}_2| = \frac{S}{p^2}\left|\frac{dS}{ds}\right|, \tag{18}$$

Moreover, colinear directions $\overline{\rho}$ and $\overline{a}_1$ have opposite senses.

We only approach case $s > 0$, $\theta < 0$ in I, in accordance with Fig. 2. Starting from equations (4), we have:

$$\overline{\tau} = \frac{\overline{v}}{v} = \frac{\dot{r}}{v}\overline{\rho} + \frac{r\dot{\theta}}{v}\overline{\varepsilon},$$

whence we make explicit $\overline{\varepsilon}$:

$$\overline{\varepsilon} = \frac{v}{r\dot{\theta}}\overline{\tau} - \frac{\dot{r}}{r\dot{\theta}}\overline{\rho}, \tag{19}$$

Entering expression (19) in the acceleration formula, we get:

$$\begin{aligned}\overline{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\overline{\rho} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\overline{\varepsilon} = \\ &= \left(\ddot{r} - r\dot{\theta}^2 - 2\frac{\dot{r}^2}{r} - \frac{\dot{r}\ddot{\theta}}{\dot{\theta}}\right)\overline{\rho} + \frac{v}{r\dot{\theta}}\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\overline{\tau},\end{aligned}$$

Taking into account equations (16) și (15), we get:

$$a_1 = \left| \ddot{r} - r\dot{\theta}^2 - 2\frac{\dot{r}^2}{r} - \frac{\dot{r}\ddot{\theta}}{\dot{\theta}} \right| = \frac{\left| 2\dot{r}^2\dot{\theta} + r\dot{r}\ddot{\theta} - r\ddot{r}\dot{\theta} + r^2\dot{\theta}^3 \right|}{r\left|\dot{\theta}\right|} = \frac{\left|\overline{v} \times \overline{a}\right|}{r\left|\dot{\theta}\right|} = \frac{\frac{v^3}{R_C}}{\frac{v}{r}p} = \frac{(pv)^2}{R_C p^3},$$

respectively

$$a_2 = \frac{\left|2\dot{r}\dot{\theta} + r\ddot{\theta}\right|}{\frac{r\left|\dot{\theta}\right|}{v}} = \frac{\left|\frac{d}{dt}\left(r^2\dot{\theta}\right)\right|}{\frac{r^2\left|\dot{\theta}\right|}{v}} = \frac{\left|\frac{d}{dt}(-pv)\right|}{p} = \frac{\left|\dot{S}\right|}{p} = \frac{\left|\frac{dS}{ds}\dot{S}\right|}{p} = \frac{v}{p}\left|\frac{dS}{ds}\right| = \frac{S}{p^2}\left|\frac{dS}{ds}\right|,$$

In the end, applying the right hand rule, we notice that vectors $\overline{v} \times \overline{a} = \left(2\dot{r}^2\dot{\theta} + r\dot{r}\ddot{\theta} - r\ddot{r}\dot{\theta} + r^2\dot{\theta}^3\right)\overline{k}$ and $-\overline{k}$ have the same sense, thus vector:

$$\overline{a}_1 = \frac{r\ddot{r}\dot{\theta} - r^2\dot{\theta}^3 - 2\dot{r}^2\dot{\theta} - r\dot{r}\ddot{\theta}}{r\dot{\theta}}\overline{\rho} = \frac{pozitiv}{negativ}\overline{\rho},$$

has opposite sense to vector $\overline{\rho}$.

5. CONCLUSIONS

Coordinates that are not orthogonal are more natural than Cartesian or polar coordinates in some settings, like the study of force problems of classical mechanics in the plane.

REFERENCES

[1]. **Iacob, C., Ştefan, G. I.; Soare, M., Dragoş, L.,** *Dicționar de mecanică*, Editura Științifică și Enciclopedică, București, 1980

[2]. **Mustafa, O. G.,** *Elemente de mecanica punctului material și a solidului rigid*, Editura Didactică și Pedagogică, București, 2006. On-line la adresa:

[3]. https://www.dropbox.com/s/5cfmp5zl3ess15m/mechanics.pdf

[4]. **Mustafa, O. G.,** *Curbe și suprafețe*

[5]. **O'Neill, B.,** *Elementary differential geometry*, Elsevier, Amsterdam, 2006