A NEW APPROACH REGARDING THE ANGULAR MOMENTUM IN CLASSICAL DYNAMICS AND SOME OF ITS CONSEQUENCES

CIPRIAN BOLOG*

Abstract: The paper introduces a new approach in defining the angular momentum for a system of particles (either continuous or discontinuous) in classical Dynamics and also shows some of its consequences regarding the theorem of the angular momentum, the computation formula for the kinetic energy and a new demonstration for the Steiner relationships.

Keywords: angular momentum, theorem of angular momentum, kinetic energy, Steriner relationships.

1. A NEW APPROACH IN DEFINING THE ANGULAR MOMENTUM

Let as consider a system of particles (either continuous or discontinuous), as shown in Fig.1., consisting of the material points A_i of mass m_i , having the mass center C, who's position vector from the reference point Q is given by:

$$\bar{r}_C = \frac{\sum m_i \bar{r}_i}{\sum m_i} \tag{1}$$

$M = \sum m_i$ being the total mass of the system.

We'll also make use of the fact that by time diffentiating a vector for which the origin and the end are mobil points (e.g. $\bar{r}_i$) we get:

$$\dot{\bar{r}}_i = \bar{v}_i - \bar{v}_Q \tag{2}$$

$$\ddot{\bar{r}}_i = \bar{a}_i - \bar{a}_Q \tag{3}$$

* *Professor, Ph D , University of Petroşani bolog_ciprian@yahoo.ro*

If our system of particles is a rigid body the equation (2) becomes:

$$\dot{\bar{r}}_i = \bar{\omega} - \bar{r}_i \tag{4}$$

We'll start from the well known definition of the angular momentum for a system of particles:

$$\bar{K}_Q = \sum \bar{r}_i \times m_i \bar{v}_i \tag{5}$$

which can be written as:

$$\bar{K}_Q = \sum m_i \bar{r}_i \times \left(\bar{v}_Q + \dot{\bar{r}}_i \right) = \left(\sum m_i \bar{r}_i \right) \times \bar{v}_Q + \sum m_i \bar{r}_i \times \dot{\bar{r}}_i$$

$$\bar{K}_Q = \bar{K}_{Q\,tr} + \bar{K}_{Q\,rot} \tag{6}$$

$$\bar{K}_{Q\,tr} = \bar{r}_C \times M\, \bar{v}_Q \tag{7}$$

$$\bar{K}_{Q\,rot} = \sum m_i \bar{r}_i \times \dot{\bar{r}}_i \tag{8}$$

If the system is a rigid body eq.(8) leads to:

$$\bar{K}_{Q\,rot} = \sum m_i \bar{r}_i \times \left(\bar{\omega} \times \bar{r}_i \right) = \sum m_i \left[\left(\bar{r}_i \cdot \bar{r}_i \right) \bar{\omega} - \left(\bar{\omega} \cdot \bar{r}_i \right) \cdot \bar{r}_i \right] \tag{9}$$

from which one could get the well known equations:

$$\begin{cases} K_{Q\,rot\,x} = J_x \omega_x - J_{xy} \omega_y - J_{xz} \omega_z \\ K_{Q\,rot\,y} = J_y \omega_y - J_{yz} \omega_z - J_{yx} \omega_x \\ K_{Q\,rot\,z} = J_z \omega_z - J_{zx} \omega_x - J_{zy} \omega_y \end{cases}$$

On the other hand, from the definition (5) and using Fig.1 we get:

$$\bar{K}_Q = \sum \left(\bar{r}_C + \bar{\rho}_i \right) \times m_i \bar{v}_i = \bar{r}_C \times \sum m_i \bar{v}_i + \sum \bar{\rho}_i \times m_i \bar{v}_i = \bar{r}_C \times \bar{H} + \bar{K}_C \tag{10}$$

where $\bar{H}$ is the linear momentum of the system and $\bar{K}_C$ is its angular momentum with respect to the mass center C. But:

$$\bar{H} = M \bar{v}_C$$

$$\bar{K}_C = \bar{K}_{C\,rot} \qquad \left(\bar{K}_{C\,tr} = \bar{\rho}_C \times M \bar{v}_C = 0 \text{ because } \bar{\rho}_C = 0 \right)$$

and thus Eqs. (10) leads ro:

$$\overline{K}_Q = \overline{r}_C \times M\overline{v}_C + \overline{K}_C = \overline{r}_C \times M\overline{v}_C + \overline{K}_{C\,rot} \tag{11}$$

Making use of the eq.(6) and (7), Eq. (11) becomes.

$$\overline{K}_{Q\,tr} + \overline{K}_{Q\,rot} = \overline{r}_C \times M\overline{v}_C + \overline{K}_{C\,rot}$$

$$\overline{K}_{Q\,rot} = \overline{r}_C \times M\overline{v}_C - \overline{r}_C \times M\overline{v}_Q + \overline{K}_{C\,rot} = \overline{r}_C \times M\dot{\overline{r}}_C + \overline{K}_{C\,rot} \tag{12}$$

If the system is a rigid body Eq. (12) becomes

$$\overline{K}_{Q\,rot} = M\overline{r}_C \times \left(\overline{\omega} \times \overline{r}_C\right) + \overline{K}_{C\,rot} \tag{13}$$

2. CONSEQUENCES UPON THE THEOREM OF THE ANGULAR MOMENTUM

By differentiating the equations (7) and (18) with respect to time we get:

$$\dot{\overline{K}}_{Q\,tr} = \dot{\overline{r}}_C \times M\overline{v}_Q + \overline{r}_C \;\; + M\dot{\overline{v}}_Q = \left(\overline{v}_C - \overline{v}_Q\right) \times M\overline{v}_Q + \overline{r}_C \times M\overline{a}_Q$$

$$\dot{\overline{K}}_{Q\,tr} = M\overline{v}_C \times \overline{v}_Q + \overline{r}_C \;\; \times M\overline{a}_Q \tag{14}$$

$$\dot{\overline{K}}_{Q\,rot} = \sum\left(m_i \dot{\overline{r}}_i \times \dot{\overline{r}}_i + m_i \overline{r}_i \times \ddot{\overline{r}}_i\right) = \sum m_i \overline{r}_i \times \left(\overline{a}_i - \overline{a}_Q\right) =$$

$$\sum \overline{r}_i \times m_i \overline{a}_i - \left(\sum m_i \overline{r}_i\right) \times \overline{a}_Q = \sum \overline{r}_i \times \overline{F}_i - M\overline{r}_C \times \overline{a}_Q = \overline{M}_Q - \overline{r}_C \times M\overline{a}_Q$$

$$\dot{\overline{K}}_{Q\,rot} = \overline{M}_Q - \overline{r}_C \times M\overline{a}_Q \tag{15}$$

From equations (6), (14) and (15) it follows that.

$$\dot{\overline{K}}_Q = \overline{M}_Q + M\overline{v}_C \times \overline{v}_Q \tag{16}$$

From (16) ant (15) we get the following two forms of the angular momentum theorem:

$$\overline{M}_Q = \dot{\overline{K}}_Q + M\overline{v}_Q \times \overline{v}_C \tag{17}$$

$$\overline{M}_Q = \dot{\overline{K}}_{Q\,rot} + \overline{r}_C \times M\overline{a}_Q \tag{18}$$

These two forms reduce to the simpler form respectively:

$$\overline{M}_Q = \dot{\overline{K}}_Q \qquad \text{if} \qquad \overline{v}_Q = 0, \quad or \quad \overline{v}_C = 0, \quad or \quad Q = C \qquad (17')$$

$$\overline{M}_Q = \dot{\overline{K}}_{Q\ rot} \ \text{ if } Q \equiv C, or \quad \overline{a}_Q = 0 \Rightarrow \quad \overline{v}_Q = 0 \quad or \quad \overline{v}_Q = \overline{C}onst, or \quad \overline{a}_Q = \lambda \overline{r}_C \quad (18')$$

I strongly suggest the use of the forms (18) or (18') of the theorem instead of the forms (17) or (17'), for two good reasons:

- The conditions for the form (18') alow a wider choice of the reference point Q than the conditions for (17') (which are included in the former ones).
- In case of the rigid body computing $\overline{K}_{Q\,rot}$ (see equations 9') is much easier than computing $\overline{K}_Q$ (see equation 6, which also requires equations 9').

In the paper [1] I simply adopted the definition (8) for the angular momentum $\overline{K}_Q$ of a rigid body, instead of definition (5). The greatest part of the treates of Mechanics (e.g. [2] and [3]) use the definition (5) for the angular momentum $\overline{K}_Q$, but the reference point Q is selected to be either a fixed point ($\overline{v}_Q = 0$) or the mass center ($Q \equiv C$). The great disadvantage of this approach is the fact that it requires two different demonstrations for the angular momentum theorem and also two different demonstrations for obtaining the computation equations (9') of the angular momentum

Using $\overline{K}_{Q\,rot}$ instead of $\overline{K}_Q$ is also important in computing the kinetic energy (see the next chapter) in case of a rigid body.

3. COMPUTATION FORMULA FOR THE KINETIC ENERGY

By definition, the kinetic energy of a system of particles is:

$$E_C = \sum \frac{1}{2} m_i v_i^2 \qquad (19)$$

The equation (19) becomes successively:

$$E_C = \sum \frac{1}{2} m_i \overline{v}_i \cdot \overline{v}_i = \sum \frac{1}{2} m_i \left(\overline{v}_Q + \dot{\overline{r}}_i \right) \cdot \left(\overline{v}_Q + \dot{\overline{r}}_i \right) =$$

$$= \frac{1}{2} \left(\sum m_i \right) \overline{v}_Q \cdot \overline{v}_Q + \sum m_i \cdot \overline{v}_Q \cdot \dot{\overline{r}}_i + \frac{1}{2} \sum m_i \cdot \dot{\overline{r}}_i \cdot \dot{\overline{r}}_i$$

$$E_C = \frac{1}{2}Mv_Q^2 + \bar{v}_Q \cdot M\dot{\bar{r}}_C + \frac{1}{2}\sum m_i \cdot \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i \tag{20}$$

In case of a rigid body the equation (20) becomes:

$$E_C = \frac{1}{2}Mv_Q^2 + M\bar{v}_Q \cdot \left(\bar{\omega} \times \bar{r}_C\right) + \frac{1}{2}\sum m_i \cdot \left(\bar{\omega} \times \bar{r}_i\right) \cdot \left(\bar{\omega} \times \bar{r}_i\right) =$$
$$= \frac{1}{2}Mv_Q^2 + \bar{\omega} \cdot \left(\bar{r}_C \times M\bar{v}_Q\right) + \frac{1}{2}\bar{\omega} \cdot \sum m_i \cdot \bar{r}_i \times \left(\bar{\omega} \times \bar{r}_i\right)$$

and using Eqs (7) and (9) we get:

$$E_C = \frac{1}{2}Mv_Q^2 + \bar{\omega} \cdot \bar{K}_{Q\,tr} + \frac{1}{2}\bar{\omega} \cdot \bar{K}_{Q\,rot} \tag{21}$$

$$E_{C\,tr} = \frac{1}{2}M\,v_Q^2 \qquad E_{C\,rot} = \frac{1}{2}\bar{\omega} \cdot \bar{K}_{Q\,rot} \tag{22}$$

$$E_{C\,mixt} = \bar{\omega} \cdot \bar{K}_{Q\,tr} = \bar{\omega} \cdot \left(\bar{r}_C \times M\bar{v}_Q\right) \tag{23}$$

$E_{C\,mixt} = 0,\ if:\ \bar{\omega} = 0, or\ Q \equiv C, or\ \bar{v}_Q = 0, or\ \ \bar{\omega} = \lambda\bar{r}_C\ , or\ \ \bar{\omega} = \lambda\bar{v}_Q\ , or\ \bar{v}_Q = \lambda\bar{r}_C\ ,$ or C is in the plane of $\bar{v}_Q$ and $\bar{\omega}$.

4. THE STEINER RELATIONSHIPS

We'll show in this paragraph how the Steiner relationships can be optained in a new fashion, using the approach of the angular momentum studied in paragraph 1.

Let us consider the equation (13) which can be written as:

$$\bar{K}_{Q\,rot} = M\left[\left(r_C^2\right)\bar{\omega} - \left(\bar{\omega} \cdot \bar{r}_C\right)\bar{r}_C\right] + \bar{K}_{C\,rot} \tag{24}$$

Using only the first of the equations (9') for $K_{Q\,rot\,x}$ and a similar one for $K_{C\,rot\,x}$, and supposing two parallel references Oxyz and Cx′y′z′, it follows from the projection of the equation (24) on the x-axis:

$$J_x\omega_x - J_{xy}\omega_y - J_{xz}\omega_z = M\left[\left(x_C^2 + y_C^2 + z_C^2\right)\omega_x - \left(\omega_x x_C + \omega_y y_C + \omega_z z_C\right)x_C\right] +$$
$$+ J_{x'}\omega_x - J_{x'y'}\omega_y - J_{x'z'}\omega_z$$

The coefficients of ω_x and of ω_y in the two sides of the previos equation being the same, we get the relationships:

$$J_x = J_{x'} + M\left(y_C^2 + z_C^2\right) \tag{25}$$

$$J_{xy} = J_{x'y'} + M\, x_C y_C \tag{26}$$

In the same manner one can obtain from the projections of the equation (24) on the y-axis and on the z-axis:

$$J_y = J_{y'} + M\left(z_C^2 + x_C^2\right) \tag{27}$$

$$J_{yz} = J_{y'z'} + M\, y_C z_C \tag{28}$$

$$J_z = J_{z'} + M\left(x_C^2 + y_C^2\right) \tag{29}$$

$$J_{zx} = J_{z'x'} + M\, z_C x_C \tag{30}$$

Equations (25)÷(30) represent the well known Steiner relationships from the Dynamics of the rigid body.

5. CONCLUSIONS

The first paragraph deals with the new approach in defining the angular momentum $\overline{K}_Q$ given by equations (6), (7) and (8). When the system of particles is a rigid body the paper shows the relationship between $\overline{K}_{Q\,rot}$ and $\overline{K}_{C\,rot}$: equation (13).

The second paragraph shows the implication of the new approach from the first paragraph in writing the angular momentum theorem. A suggestion of using an alternate form of the theorem is presented and the reasons for doing so.

The third paragraph deals with the computational formula for the kinetic energy of a system, showing the usefulness of the equations (6), (7) and (8) in this formula.

The fourth paragraph presents a new demonstration of the Steiner relationships from the Dynamics of the rigid body, using the equation (13) of the first paragraph.

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