ON SOME MATHEMATICAL ECONOMIC MODELS SOLVED BY THE METHOD OF LAGRANGE MULTIPLIERS

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ABSTRACT: *Many of the economic and technical problems can be mathematically modelled and the mathematical model leads to solving some nonlinear equations or finding of some extreme values. One of the most popular methods of solving is the so-called Lagrange's multipliers method. This paper aims to present two economic models in which a minimum and a maximum requirement arise and which can be solved by the mentioned method.*

KEY WORDS: *Lagrange's multipliers method, static optimum model of the consumer.*

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1. INTRODUCTION

Let us consider $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ and the system of $p < n$ equations:

$$\begin{cases} F_1(x_1, x_2, \dots, x_n) = 0 \\ F_2(x_1, x_2, \dots, x_n) = 0 \\ \dots \\ F_p(x_1, x_2, \dots, x_n) = 0 \end{cases} \tag{1}$$

with $F_1, F_2, \dots, F_p: A \subseteq \mathbb{R}^n \to \mathbb{R}$. Let us also consider that $A_0 \subseteq A$ is the set of the system solution denoted by (1).

We will continue to assume that $f \in C^2(A)$ and that $F_1, F_2, \dots, F_p \in C^1(A)$ which means that there are partial derivatives up to the second order for f and first-order partial derivatives for $F_1, F_2, \dots, F_p$ and all of these partial derivatives are continuous. We will also remind some of the mathematical notions that will continue to be used.

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Definition 1 *We say that point $x_0 \in A$ is a stationary point of function f if*

$$\begin{cases} \frac{\partial f}{\partial x_1}(x_0) = 0 \\ \frac{\partial f}{\partial x_2}(x_0) = 0 \\ \cdots \\ \frac{\partial f}{\partial x_n}(x_0) = 0 \end{cases}.$$

Definition 2 *The extreme points of f when $(x_1, x_2, \ldots, x_n)$ is through the set of points A_0 of the solution of the system (1) are called the extremes of the function f conditioned by the system (1). The stationary points of f when $(x_1, x_2, \ldots, x_n)$ is through the set of points A_0 of the solution of the system (1) are called the stationary points of the function f.*

The method of Lagrange multipliers consists in the next steps:

1) We have to create the auxiliary function of Lagrange

$$L: \mathbb{R}^{n+p} \to \mathbb{R}, L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_p) =$$
$$= f(x_1, x_2, \ldots, x_n) + \lambda_1 F_1(x_1, x_2, \ldots, x_n) + \lambda_2 F_2(x_1, x_2, \ldots, x_n) + \cdots + \lambda_p F_p(x_1, x_2, \ldots, x_p);$$

2) We solve the system:

$$\begin{cases} \frac{\partial L}{\partial x_1}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_p) = 0 \\ \frac{\partial L}{\partial x_2}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_p) = 0 \\ \cdots \\ \frac{\partial L}{\partial x_n}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_p) = 0 \\ \frac{\partial L}{\partial \lambda_1}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_p) = 0 \\ \frac{\partial L}{\partial \lambda_2}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_p) = 0 \\ \cdots \\ \frac{\partial L}{\partial \lambda_p}(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_p) = 0 \end{cases}; \quad (2)$$

3) Let $(x_1{}^0, x_2{}^0, \ldots, x_n{}^0, \lambda_1{}^0, \lambda_2{}^0, \ldots, \lambda_p{}^0)$ be a solution of the system (2), it means this point is a stationary point for L. It follows that $x^0 = (x_1{}^0, x_2{}^0, \ldots, x_n{}^0)$ is a stationary point for f . The extreme points of the function f are included between stationary points of the function f ;
4) Let us consider the next value:

$$E = \frac{\partial^2 L}{\partial x_1{}^2}(x_0)(dx_1)^2 + \frac{\partial^2 L}{\partial x_2{}^2}(x_0)(dx_1)^2 + \cdots + \frac{\partial^2 L}{\partial x_n{}^2}(x_0)(dx_n)^2 + \cdots +$$
$$+2\frac{\partial^2 L}{\partial x_1 \partial x_2}(x_0)dx_1 dx_2 + \cdots + 2\frac{\partial^2 L}{\partial x_{n-1}\partial x_n}dx_{n-1}dx_n \quad (3)$$

If $E > 0$ then $x^0 = (x_1{}^0, x_2{}^0, \ldots, x_n{}^0)$ is a conditioned minimum point and if $E < 0$ then $x^0 = (x_1{}^0, x_2{}^0, \ldots, x_n{}^0)$ is a conditioned maximum point.

2. THE STATIC OPTIMUM MODEL OF THE CONSUMER

We start from the next data:

a) $x_1, x_2, \dots, x_n$ are the amount of the consumer goods required by the consumer;
b) U is the function of utility, so we shall have $U = U(x_1, x_2, \dots, x_n)$;
c) $p_1, p_2, \dots, p_n$ represents the unit price of consumer goods;
d) A is a fixed budget.

The problem that we have to solve is to find the maximum function taking care of the given data, that means to solve the next optimization problem:

$$\text{(P)} \begin{cases} maxU(x_1, x_2, \dots, x_n) \\ \sum_{i=1}^{n} x_i\, p_i = A \\ x_i \geq 0 \end{cases} \tag{4}$$

The method of solving consist initially in construction of the Lagrange function $L: \mathbb{R}^{n+1} \to \mathbb{R}$, given by:

$$L(x_1, x_2, \dots, x_n, \lambda) = f(x_1, x_2, \dots, x_n) + \lambda \left(\sum_{i=1}^{n} p_i x_i - A \right)$$

We will find the stationary points of the Lagrange function by solving the next system:

$$\begin{cases} \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} = 0 \\ \dots \\ \frac{\partial L}{\partial x_n} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial U}{\partial x_1} + \lambda x_1 = 0 \\ \frac{\partial U}{\partial x_2} + \lambda x_2 = 0 \\ \dots \\ \frac{\partial U}{\partial x_n} + \lambda x_n = 0 \\ p_1 x_1 + p_2 x_2 + \dots + p_n x_n - A = 0 \end{cases} \tag{5}$$

A general solution of the system (5) cannot be obtained without knowing an analytical expression of *U*. However, from (4) we can deduce that we can obtain

$$\lambda = \frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \dots = \frac{\frac{\partial U}{\partial x_n}}{p_n}$$

that means that from an economic point of view, it can be seen that the ratios between the marginal utilities in relation to the quantity of consumer goods and the prices of these goods are constant.

Without insisting on the demonstration it can be shown that the solution, if it exists is unique and is additionally the maximum as the problem requires. It means that, taking care of (3), we will have that $E > 0$.

Below we will present a particular case of this type of problem.

Particularly case

We will still solve the problem denoted by (4) for a Cobb-Douglass utility function that has the general shape:

$$U(x_1, x_2, \dots, x_n) = a{x_1}^{\alpha_1}{x_2}^{\alpha_2} \dots {x_n}^{\alpha_n} \tag{6}$$

where a means the scale factor and $\alpha_1, \alpha_2, \dots, \alpha_n > 0, \sum_{i=1}^{n} \alpha_i$.

The Lagrange function in this particular case is $L: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$,

$$L(x_1, x_2, \dots, x_n, \lambda) = a{x_1}^{\alpha_1}{x_2}^{\alpha_2} \dots {x_n}^{\alpha_n} + \lambda(\sum_{i=1}^{n} p_i x_i - A).$$

The system (4) becomes

$$\begin{cases} \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} = 0 \\ \dots \\ \frac{\partial L}{\partial x_n} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} a\alpha_1{x_1}^{\alpha_1 - 1}{x_2}^{\alpha_2} \dots {x_n}^{\alpha_n} + \lambda x_1 = 0 \\ a\alpha_2{x_1}^{\alpha_1}{x_2}^{\alpha_2 - 1} \dots {x_n}^{\alpha_n} + \lambda x_2 = 0 \\ \dots \\ a\alpha_n{x_1}^{\alpha_1}{x_2}^{\alpha_2} \dots {x_n}^{\alpha_n} + \lambda x_n = 0 \\ p_1x_1 + p_2x_2 + \dots + p_nx_n - A = 0 \end{cases}$$

We find that

$$\begin{cases} a\alpha_1{x_1}^{\alpha_1 - 1}{x_2}^{\alpha_2} \dots {x_n}^{\alpha_n} = -\lambda x_1 \\ a\alpha_2{x_1}^{\alpha_1}{x_2}^{\alpha_2 - 1} \dots {x_n}^{\alpha_n} = -\lambda x_2 \\ \dots \\ a\alpha_n{x_1}^{\alpha_1}{x_2}^{\alpha_2} \dots {x_n}^{\alpha_n} = -\lambda x_n \\ p_1x_1 + p_2x_2 + \dots + p_nx_n - A = 0 \end{cases}.$$

Sharing the first equation to the next equations we find:

$$\begin{cases} \frac{\alpha_1}{\alpha_2}\frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{\alpha_2}{\alpha_1}\frac{p_1}{p_2}x_1 \\ \frac{\alpha_1}{\alpha_3}\frac{x_3}{x_1} = \frac{p_1}{p_3} \Rightarrow x_3 = \frac{\alpha_3}{\alpha_1}\frac{p_1}{p_3}x_1 \\ \dots \\ \frac{\alpha_1}{\alpha_n}\frac{x_n}{x_1} = \frac{p_1}{p_n} \Rightarrow x_n = \frac{\alpha_2}{\alpha_1}\frac{p_1}{p_n}x_1 \\ p_1x_1 + p_2x_2 + \dots + p_nx_n - A = 0 \end{cases}$$

Introducing the first $(n-1)$ equations of the system in the las one we will fiind that

$$p_1x_1 + p_1x_1\frac{\alpha_2}{\alpha_1} + p_1x_1\frac{\alpha_3}{\alpha_1} + \dots + p_1x_1\frac{\alpha_n}{\alpha_1} = A$$

which implies

$$p_1x_1\left(\frac{\alpha_1 + \alpha_2 + \dots + \alpha_n}{\alpha_1}\right) = A.$$

Taking care that $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ we find $x_1 = A_1\frac{\alpha_1}{p_1}$.

The optimal solution will be in this case:

$$x_1^* = A\frac{\alpha_1}{p_1}, x_2^* = A\frac{\alpha_2}{p_2}, \dots, x_n^* = A\frac{\alpha_n}{p_n} \quad (7)$$

which means that for this optimal consumption if we take care of (5) that we will find the maximum value of utility as being

$$U(x_1^*, x_2^*, \dots, x_n^*) = aA^{\alpha_1+\alpha_2+\dots+\alpha_n}\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\dots\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\dots\left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}$$

and also taking care that $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ we find

$$U(x_1^*, x_2^*, \dots, x_n^*) = aA\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\dots\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\dots\left(\frac{\alpha_n}{p_n}\right)^{\alpha_n}. \quad (8)$$

Numerical example

Assuming we have three consumer goods having the unit prices of 15, 20 and 25 monetary units and a fixed budget of 100 unitary units. Let us solve the problem in terms of Cobb-Douglass type utility having $U(x_1, x_2, x_3) = 12{x_1}^{0.4}{x_2}^{0.3}{x_3}^{0.3}$.

Solution

Taking care of (7) and (8) we get

$$x_1^* = 100\frac{0.4}{15} \approx 2.66, x_2^* = 100\frac{0.3}{20} = 1.5, x_3^* = 100\frac{0.3}{25} = 1.2$$

and the maximum required by the problem

$$U(x_1^*, x_2^*, x_3^*) = 12 \cdot 100 \cdot \left(\frac{0.4}{15}\right)^{0.4} \cdot \left(\frac{0.3}{20}\right)^{0.3} \cdot \left(\frac{0.3}{25}\right)^{0.3} \approx 21{,}19$$

3. THE MINIMAL PROBLEM IN SQUARE LINEAR COMBINATION

Let us consider the solving of problem of finding the minimum of the function $U(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \propto_i (x_i - a_i)^2, \alpha_i > 0, i = 1, \dots, n$, this function model being quite common in many applications and having the initial conditions:

$$\sum_{i=1}^{n} x_i\, p_i = A, x_i \geq 0, i = 1, \dots, n$$

In other words, we have to solve the problem:

$$\text{(P)} \begin{cases} minU(x_1, x_2, \dots, x_n) = \min(\sum_{i=1}^{n} \propto_i (x_i - a_i)^2) \\ \sum_{i=1}^{n} x_i\, p_i = A \\ x_i \geq 0 \end{cases} \quad (9)$$

In this case the Lagrange function is given by $L: \mathbb{R}^{n+1} \to \mathbb{R}$,

$$L(x_1, x_2, \dots, x_n, \lambda) = \sum_{i=1}^{n} \alpha_i (x_i - a_i)^2 + \lambda(\sum_{i=1}^{n} p_i x_i - A).$$

In this case the system (2) becomes

$$\begin{cases} \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} = 0 \\ \dots \\ \frac{\partial L}{\partial x_n} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 2\alpha_1(x_1 - a_1) + \lambda p_1 = 0 \\ 2\alpha_2(x_2 - 2) + \lambda p_2 = 0 \\ \dots \\ 2\alpha_n(x_n - a_n) + \lambda p_n = 0 \\ p_1x_1 + p_2x_2 + \dots + p_nx_n - A = 0 \end{cases}$$

which represents a linear system of $(n + 1)$ equations whose solving is usually done by Cramer's rule.

It is easily to demonstrate that

$$E = 2(\alpha_1 + \alpha_2 + \dots + \alpha_n) > 0$$

which means that, in the specified conditions, the problem admits the required solution, in other words, there is a minimum required of the problem.

Below we will present a hypothetical example of an application requesting the determination of a minimum expenditure under the conditions of a fixed expenditure budget. The solution will be also based on the Lagrange multipliers method.

Example

A manager uses two work of team for a particular type of work, the cost per hour pf the two teams being 50 and 60 units respectively. For practical reasons, it is known that the fatigue accumulated by each team, directly dependent on the sum of expressions squares $(x - 1{,}5)$ and $(y - 1)$ respectively, where x and y represent the number of hours performed by the two teams. There is the question of determining of a minimum fatigue of the teams when the spending budget is, for example, 1000 monetary units and, impolitely, of the numbers of hours to be performed by each team has to carry out to produce the required production under the existing budget at a minimal effort.

Solution

In this case we talk about the question of determining the minimum of the function

$$\mathrm{f}: \mathbb{R}^2 \to \mathbb{R}, \mathrm{f(x, y)} = (\mathrm{x} - 1{,}5)^2 + (\mathrm{y} - 1)^2 = \mathrm{x}^2 + \mathrm{y}^2 - 3\mathrm{x} - 2\mathrm{y} + 3{,}25$$

having the restrictions $50x + 60y = 1000$.

That means that we have the link function $: \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = 50x + 60y - 1000$.

In other words, we will have to solve the problem:

$$\begin{cases} maxf(x,y) = max(\mathrm{x}^2 + \mathrm{y}^2 - 3\mathrm{x} - 2\mathrm{y} + 3{,}25) \\ 50x + 60y = 1000 \\ x, y \geq 0 \end{cases}$$

We will build the Lagrange function

$$L: \mathbb{R}^3 \to \mathbb{R}, L(\mathrm{x},\mathrm{y},\lambda) = \mathrm{f}(\mathrm{x},\mathrm{y}) + \lambda F(\mathrm{x},\mathrm{y}) =$$
$$= x^2 + y^2 - 3x - 2y + 3{,}25 + \lambda(50x + 60y - 1000)$$

Next we will determine the stationary points of L:

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 \Rightarrow 2x - 3 + 50\lambda = 0 \\ \dfrac{\partial L}{\partial y} = 0 \Rightarrow 2y - 2 + 60\lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \Rightarrow 50x + 60y - 1000 = 0 \end{cases}$$

We obtain as the stationary point the approximate solution $\left(8.6, 9.5, -\frac{17}{60}\right)$.

We still have

$$dF = 0 \Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \Rightarrow 50dx + 60dy = 0 \Rightarrow dx = -\frac{5}{6}dy \qquad (10)$$

Also we have to evaluate the value of *E:*

$$E = \frac{\partial^2 L}{\partial x^2}(dx)^2 + \frac{\partial^2 L}{\partial y}(dy)^2 + 2\frac{\partial^2 L}{\partial x \partial y}dxdy = 2(dx)^2 + 2(dy)^2 = \frac{122}{25}(dy)^2 > 0$$

if we take care of (10).

So the point $\left(8.6, 9.5, -\frac{17}{60}\right)$ is a minimally conditioned one which means means that in order to execute works by amount of the allocated budget, in this case about 1000 unitary units with a minimum effort, the first team will have to work 8.6 hours and the second 9.5 hours.

4. CONCLUSIONS

It is easily to observe the flexibility of the Lagrange multipliers method taking care that it can model both maximum and minimum problem. So, in the case of the problem of the statistic optimum model of the consumer a maximum problem is solved meanwhile in the case of the proposed hypothetical problem of getting a minimum effort for two working teams in the context of a work with a fixed budget a minimum one is solved.

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