ON SOME MODELS OF LINEAR OPTIMIZATION WITH APPLICATIONS

CĂTĂLIN-ILIE MITRAN

ABSTRACT: *The purpose of this paper is to present an economic model of profit maximization in case of a factory that has a number of equipments that must produce a certain number of workpieces in terms of time and profit specified. Using two mathematical models for solving linear optimization problems will be solved a particular case of the mentioned economic problem. One of the method is based on graphics resolution and the other on the simplex algorithm. First of all will be presented the data of the problem with a particular case for solving and after these a short presentation of the two above mentioned methods followed by the itself solve of the problem using these methods and the results analysis.*

KEY WORDS: *linear optimization problem, simplex algorithm, graphics method, polygon of the solutions.*

JEL CLASSIFICATION: *C61.*

1. INTRODUCTION

A company has a number of m production equipments $M_1, M_2, \dots, M_m$ which must process n different types of workpieces $P_1, P_2, \dots, P_n$. The time expressed in minutes required during processing a workpiece is different depending on the type of equipment and the type of workpiece. We know the next data:

$t_{ij} -$ the $P_j, j = 1, \dots, n$ workpiece processing time required by the equipment $M_i, i = 1, \dots, m$ for one month;
$c_j -$ the capacity of processing of the equipment $M_i, i = 1, \dots, m$ for one month;
$b_j -$ the benefit expressed in monetary units obtained from the $P_j, j = 1, \dots n$ workpiece;
$x_j -$ the number of workpieces that have to be processed for a month.

The problem data are given by the following table:

P_i \ M_j	P_1	P_2	…	P_n	c_i
M_1	t_{11}	t_{12}	…	t_{1n}	c_1
M_2	t_{21}	t_{22}	…	t_{2n}	c_2
…	…	…	…	…	…
M_m	t_{m1}	t_{m2}	…	t_{m2}	c_m
b_j	b_1	b_2	…	b_n	

We can express and solve the model by using the next linear optimization problem:

$$\max\,(b_1x_1 + b_2x_2 + \cdots + b_nx_n)$$
$$\begin{cases} t_{11}x_1 + t_{12}x_2 + \cdots + t_{1n}x_n \le c_1 \\ t_{12}x_2 + t_{22}x_2 + \cdots + t_{2n}x_n \le c_2 \\ \ldots \\ t_{m1}x_1 + t_{m2}x_2 + \cdots + t_{mn}x_n \le c_m \\ x_1, x_2, \ldots, x_n \ge 0 \end{cases} \quad (1)$$

Application A company has three types of equipment denoted by M_1, M_2, M_3 having to process two types of workpieces denoted by P_1 and P_2. Using the data from the table:

P_i \ M_j	P_1	P_2	c_i
M_1	11	9	9900
M_2	7	12	8400
M_3	6	16	9600
b_j	90	100	

(2)

we must find the benefit which can be obtained.

Definition 1 We name a linear optimization problem in two variables the next problem:

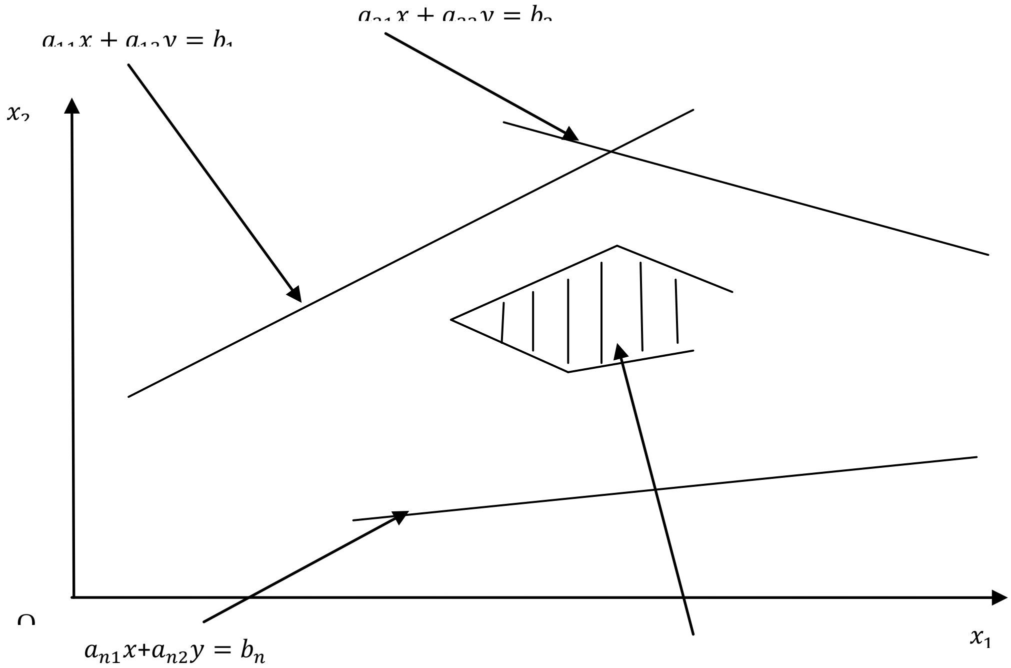
$$\max(\min)\,(C_1x + C_2y)$$
$$\begin{cases} a_{11}x + a_{12}y \le (\ge) b_1 \\ a_{21}x + a_{22}y \le (\ge) b_2 \\ \ldots \\ a_{n1}x + a_{n2}y \le (\ge) b_n \\ x, y \ge 0 \end{cases} \quad (3)$$

where $C_1x + C_2y$ are named the eficiency function and the system of equalities

$$\begin{cases} a_{11}x + a_{12}y \leq (\geq) b_1 \\ a_{21}x + a_{22}y \leq (\geq) b_2 \\ \quad \dots \\ a_{n1}x + a_{n2}y \leq (\geq) b_n \\ \quad x, y \geq 0 \end{cases}$$

are named the restriction of the linear optimization problem.

In the standard bidimensional xOy plan of coordinates the restrictions $a_{11}x + a_{12}y \leq (\geq) b_1$, $a_{21}x + a_{22}y \leq (\geq) b_2$,…, $a_{n1}x + a_{n2}y \leq (\geq) b_n$ are half-planes and the intersection of all half-planes represents a convex polygon. The maximum or the minimum of problem (1) will be obtained in one of the in the vertices of the polygon.



The polygon of the solutions

Definition 2 We name a standard linear optimization problem the next problem:

$$\max(\min)\,(\mathrm{C_1x_1} + \mathrm{C_2x_2} + \cdots + \mathrm{C_nx_n})$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ x_1, x_2, \dots, x_n \geq 0 (m < n) \end{cases} \tag{4}$$

where $C_1x_1 + C_2x_2 + \cdots + C_nx_n$ are named the eficiency function and the system of equalities

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ x_1, x_2, \dots, x_n \geq 0 (m < n) \end{cases}$$

are named the restriction of the linear optimization problem.

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

are named the matrix of the restriction and we suppose that $rank(A) = m$. It follows that it exists a matrix B called base so that

$$B = \begin{pmatrix} a_{1i_1} & a_{1i_2} & \dots & a_{1i_m} \\ a_{2i_1} & a_{2i_2} & \dots & a_{2i_m} \\ \dots & \dots & \dots & \dots \\ a_{mi_1} & a_{mi_2} & \dots & a_{mi_m} \end{pmatrix}$$

called base so that det $(B) \neq 0$.

Solving the problem lies in the algorithm simplex. A table of this algorithm is of the form:

VB	VVB	x_1	x_2	…	x_n
x_{i_1} x_{i_2} … x_{i_m}	$B^{-1}b$	$B^{-1}a_1$	$B^{-1}a_2$	…	$B^{-1}a_n$
	Z	$Z_1 - c_1$	$Z_2 - c_2$	…	$Z_n - c_n$

where VB means *the vectors of the base* and VVB *the values of the vectors of the base* and the values $Z, Z_1, \dots, Z_n$ which are given by $Z = C_B B^{-1} b$, $Z_1 = C_B B^{-1} a_1$,…, $Z_n = C_B B^{-1} a_n$ have the next semnification:

Z- the variable for calculus of maximum or minimum of the problem;

$Z_1, \dots, Z_n$- variables that gives us the stopping condition. In the case of the problem of maximum we will stop then all the variables $Z_1, \dots, Z_n$ will be positive and in the case of the of the problem of minimum we will stop then all the variables $Z_1, \dots, Z_n$ are negative.

In the case that the stopping condition is not verified we will build a new table taking care of the next:

1) If the problem ia a maximum one and between the variables $Z_1, \dots, Z_n$ are someones negative we will select the smallest of these values which is Z_k to suppose and we will got o 3);
2) If the problem is a minimum one and between the variables $Z_1, \dots, Z_n$ are someones positive we will select the biggest of these values which is Z_k to suppose and we will go to 3);
3) For the choosing of the pivot element we will calculate the minimum of the positive reports that can be formed using elements from the called VVB column at the numerator and the corresponding column of the variable x_k . Let's suppose that this minimum is obtained for the corresponding vector of the base $x_l, l \in \{i_1, \dots, i_m\}$;
4) The vectors belonging to the column denoted by VVB excepting x_k will still remain in the next table, the vector x_k will replace the vector x_l, the columns of the variables $x_{i_1}, x_{i_2}, \dots, x_{i_m}$ excepting x_l will remain unchanged, the column of the vector x_k will be the old column of the variable x_l , the pivot line will be divided by the pivot element and the remaining elements will be calculate using the rectangle rule reported to the pivot element.
5) The rule of the rectangle of an reported to the pivot element is given by the next calculation rule:

$x_{\alpha k} \dots \dots x_{\alpha\beta}$
$\dots \qquad \dots$ (the rectangle made by four elements)
$x_{lk} \dots \dots . x_{l\beta}$

$$x_{\alpha\beta} \leftarrow x_{\alpha\beta} - \frac{x_{\alpha k} \cdot x_{l\beta}}{x_{lk}} .$$

Taking care of (1) and (2) the mathematical problem is given by:

$$max(90x_1 + 100x_2)$$
$$\begin{cases} 11x_1 + 9x_2 \leq 9900 \\ 7x_1 + 12x_2 \leq 8400 \\ 6x_1 + 16x_2 \leq 9600 \\ \quad x_1, x_2 \geq 0 \end{cases} \qquad (5)$$

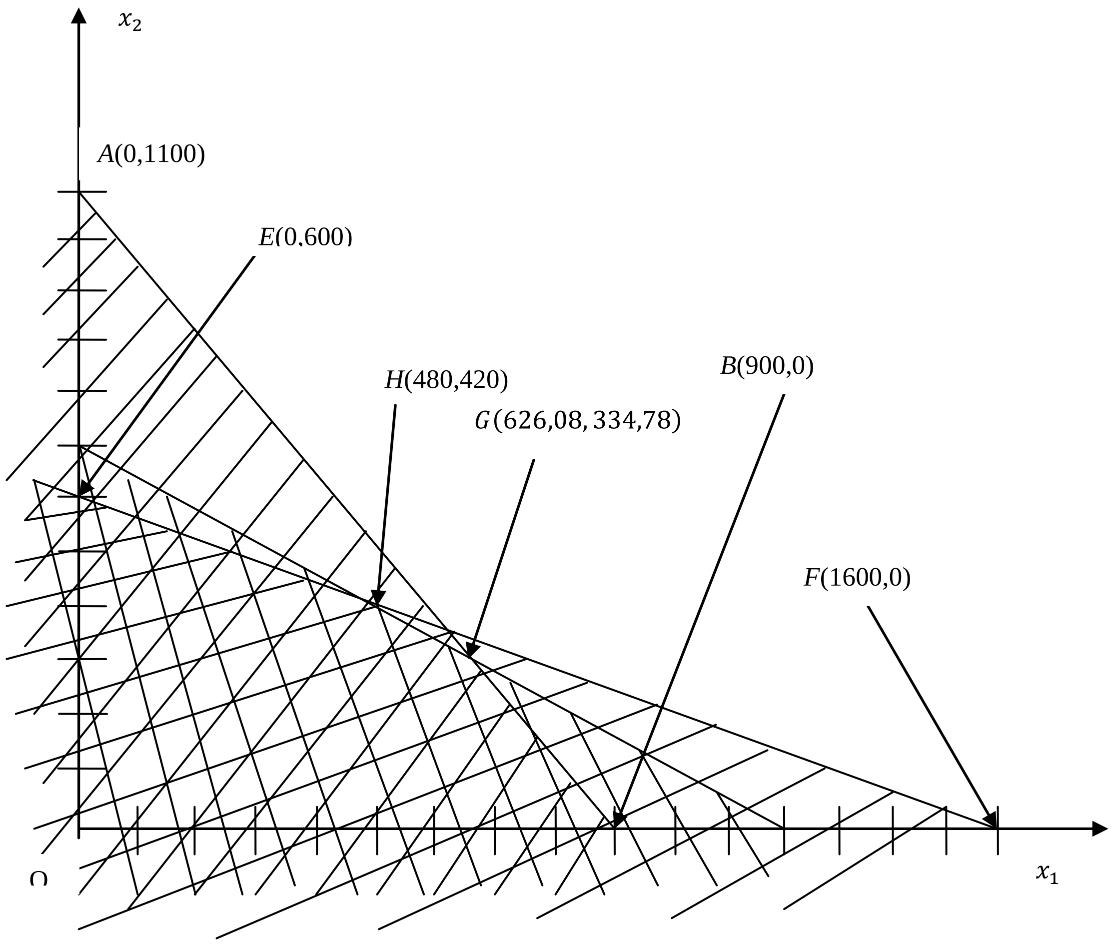
2. METHOD I (GRAPHICS METHOD)

In order to represent the three half-planes we have to represent first the associated lines. To represent these lines we have is enough to determine two points belonging to these lines. We will have:

$11x_1 + 9x_2 = 9900$
$x_1 = 0 \Rightarrow 9x_2 = 9900 \Rightarrow x_2 = 1100$. Corresponding point $A(0,1100)$.
$x_2 = 0 \Rightarrow 11x_1 = 9900 \Rightarrow x_1 = 900$. Corresponding point $B(900,0)$.

$7x_1 + 12x_2 = 8400$
$x_1 = 0 \Rightarrow 12x_2 = 8400 \Rightarrow x_2 = 700$. Corresponding point $C(0{,}700)$.
$x_2 = 0 \Rightarrow 7x_1 = 8400 \Rightarrow x_1 = 1200$. Corresponding point $D(1200{,}0)$.
$6x_1 + 16x_2 = 9600$
$x_1 = 0 \Rightarrow 16x_2 = 9600 \Rightarrow x_2 = 600$. Corresponding point $E(0{,}600)$.
$x_2 = 0 \Rightarrow 6x_1 = 9600 \Rightarrow x_1 = 1600$. Corresponding point $F(1600{,}0)$.

All the three half-planes are located below the associated lines and their intersection in the field $x \geq 0, y \geq 0$ is given by the polygon *EOBGH*. The maximum requested by the problem will be obtained in one of these vertices. So we have to calculate the values of function $f(x_1, x_2) = 90x_1 + 100x_2$ in all these vertices and their to choose the maximum value of them.



We have for the coordinates of *G* to solve the system: $\begin{cases} 11x_1 + 9x_2 = 9900 \\ 7x_1 + 12x_2 = 8400 \end{cases}$.

We get $G(626{,}08, 334{,}78)$.

We have for the coordinates of H to solve the system: $\begin{cases} 7x_1 + 12x_2 = 8400 \\ 6x_1 + 16x_2 = 9600 \end{cases}$.

We get $H(480, 420)$.

We will calculate the value of the function $f(x_1, x_2) = 90x_1 + 100x_2$ in every point $E(0,600)$, $O(0,0)$, $B(900,0)$, $G(626{,}08,\ 334{,}78)$ and $H(480,420)$. We get:

$f(0,600) = 60000$
$f(0,0) = 0$
$f(900,0) = 81000$
$f(626{,}08; 334{,}78) = 89819{,}2$
$f(480,420) = 85200$

We get $max(90x_1 + 100x_2) = 89819{,}2$ in $G(626{,}08, 334{,}78)$. But if we take care that x_1 and x_2 which represent a type of workpieces must be integers we chose for x_1 and x_2 the values $x_1 = 626$ and $x_2 = 334$. In this case the maximum requested by the problem will be $90 \cdot 626 + 100 \cdot 334 = 89740$.

3. METHOD II (SIMPLEX ALGORITHM METHOD)

Starting from the initial problem by adding the supplimentars variables we get the equivalent problem:

$$max(90x_1 + 100x_2)$$
$$\begin{cases} 11x_1 + 9x_2 + x_3 = 9900 \\ 7x_1 + 12x_2 + x_4 = 8400 \\ 6x_1 + 16x_2 + x_5 = 9600 \\ \quad x_1, x_2, \dots, x_5 \geq 0 \end{cases}$$

In our case we have the next elements:

$$A = \begin{pmatrix} 11 & 9 & 1 & 0 & 0 \\ 7 & 12 & 0 & 1 & 0 \\ 6 & 16 & 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 9900 \\ 8400 \\ 9600 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$C = (90\ \ 100\ \ 0\ \ 0\ \ 0),\ C_B = (0\ \ 0\ \ 0)$$

The first table will be:

VB	VVB	x_1	x_2 ↓	x_3	x_4	x_5
x_3	9900	11	9	1	0	0
x_4	8400	7	12	0	1	0
← x_5	9600	6	16	0	0	1
	0	-90	-100	0	0	0

Due to the fact that between the elements $z_i - c_i, i = 1, \dots, n$ we have negative values we will got o the next table:

VB	VVB	x_1	x_2	x_3	x_4	x_5
x_3	4500	61/8	0	1	0	-9/16
x_4	1200	5/2	0	0	1	-3/4
x_2	600	3/8	1	0	0	1/16
	0	-105/2	0	0	0	25/4

Also, we have the value $z_1 - c_1 < 0$ and we have to got o the third table:

VB	VVB	x_1	x_2	x_3	x_4	x_5
x_3	840	0	0	1	-61/20	69/40
x_1	480	1	0	0	2/5	-3/10
x_2	420	0	1	0	-3/20	7/40
	86200	0	0	0	21	-19/2

And again, due to the fact that $z_5 - c_5 < 0$ we have to got o the fourth table:

VB	VVB	x_1	x_2	x_3	x_4	x_5
x_5	486,95	0	0	40/69	-121/69	1
x_1	626,08	1	0	4/23	-3/23	0
x_2	334,78	0	1	-7/69	11/69	0
	89819,2	0	0	308/69	2608/69	0

Again we get the same result 89819,2 for maximum required by the problem for the values $x_1 = 626{,}08$ and $x_2 = 334{,}78$ and of course, taking care of that x_1 and x_2 must be integers and choosing for x_1 and x_2 the values $x_1 = 626$ and $x_2 = 334$ we get the value $90 \cdot 626 + 100 \cdot 334 = 89740$ for the maximum requested by the problem.

4. CONCLUSIONS

The results obtained, of course identical, by the two methods of linear optimization show us the importance and the power and the applicability of the mathematical methods in simulation and solving of some types of technical and economic problems. In our case, starting from some technical data, the mathematical models sole an concrete economic problem of determining a maximum profit.

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