IDENTIFYING THE PARAMETERS OF THE MATHEMATICAL EXPENDITURE SYSTEM MODEL

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ABSTRACT: *This chapter describes an optimum regulation model for the public expenditures system in Romania. The aim of this work is to design an optimal control system of public expenditures in Romania. It contains an offline identification of the total public expenditures system in Romania for a timespan of 15 years. The total public expenditures system is a MISO type one (Multiple Input – Single Output) and is identified by the use of the lowest foursquare applied on an OE (Output Error) type model.*

KEY WORDS: *public expenditures, Gross Domestic Product, stochastic models, control system.*

JEL CLASSIFICATION: *E60, H50.*

1. INTRODUCTION

At present times, Romania does not have a well defined and optimised control system through which it can regulate public expenditures. Even though there have been released a number of studies on the subject, a general consensus has still not been reached so that scientific research could offer clear public expenditure optimisation models. The effectiveness of public expenditures could have a significant impact on the GDP, making this endeavour a highly desired objective for every country, including ours. Romania's mathematical public expenditure model is presented in the first part of this work. Based on this model for a certain reference amount imposed on public expenses, one can determine the composing elements of the input vector from the mathematical econometric system of Romania's public expenditures. It describes a regulating system for total public expenditures in Romania, utilizing a linear

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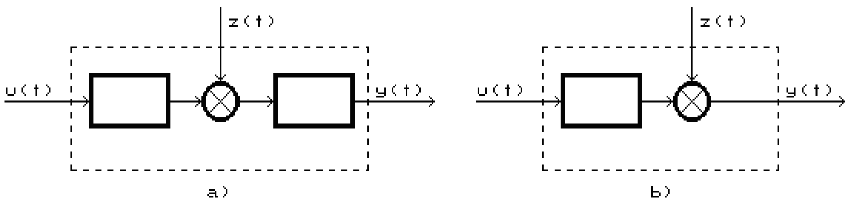
foursquare performance indicator. The Romanian total public expenditures system is characterised by the use of the smaller foursquare method applied on an autoregressive type Output Error model.

2. INPUT – OUTPUT STOCHASTIC MODELS

In this paragraph are being described the most utilised mono-variable stochastic models specific for SISO (Single Input – Single Output) economic systems. We mention here the auto-regressive (AR) models of types: MA (Moving Average), ARMA (Auto-regressive and moving average), ARX (auto-regressive controlled or with exogenous measures), OE (Output Error), or Box-Jenkins. There is also a mention about the multivariable stochastic model specific to MIMO (Multiple Input -Multiple Output) economic systems.

An economic system can be represented as a bloc scheme like the one hereunder in Figure 1.



Source: Landau, I.D., Identificarea și comanda sistemelor, Editura Tehnică, București, 1997

Figure 1. Bloc scheme of an economic system

As a principle, the perturbation can act anywhere inside the economic process but if the system is linear, it can be transferred towards the exit of the system.

In the case when the perturbation *z(t)* (noise) slightly influences the exit measurement *y(t)*, it can be ignored in the process of economic control; though when the influence is greater one must also take into consideration the path through which the perturbation is channelled towards the exit, in other words one also should require the mathematical model of the noise path.

In this case, the growth evolution can be determined if we know the models of the two paths (of control and noise), the entry signal *u(t)* and the statistical characteristics of noise *z(t).* If the perturbation is a random process having rational spectral density, as per the theory of spectral factorisation, *z(t)* can be interpreted as being the exit of a rational stable filter of minimal phase at the entry of which is applied the white noise *e(t)* (Figure 2).

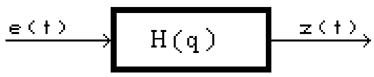


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Figure 2. Bloc scheme of white noise filter

If *H(q)* is the transfer function of this filter, then $z(t)=H(q)\cdot e(t)$. In this situation, a slightly different model, but nevertheless compliant with Figure 1.1.b is:

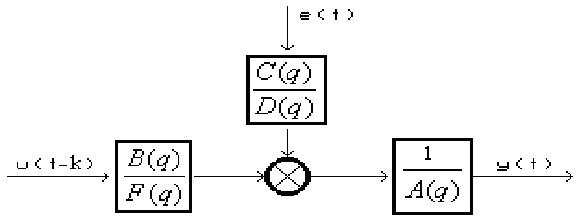
$$y(t) = G(q)\cdot u(t) + H(q)\cdot e(t) \tag{1.1}$$

The filters *G(q), H(q)*, are functions which depend on variable q. Different forms of *G(q), H(q)* lead to different models. The most general form, compliant to Figure 1.1.a, is :

$$\begin{cases} A(q)\cdot y(t) = \dfrac{B(q)}{F(q)}\cdot u(t-k) + \dfrac{C(q)}{D(q)}\cdot e(t) \\ M\left[e^2(t)\right] = \lambda^2 \end{cases} \tag{1.2}$$

Where we have : M is the statistical average, λ is noise dispersion, k is dead time expressed in the number of sampling periods, *e(t)* is model error, *t* is normalised time (real time divided by sampling period), *t* takes values from the whole numbers group $t \in \mathbf{Z}$, u(t) is the entry value at moment t , y(t) is the exit value at moment t , q^{-1} is the delay operator by one degree $q^{-1}\cdot u(t) = u(t-1)$; $q^{-k}\cdot u(t) = u(t-k)$.

The model described in equation (1.2) is also called ***theta model*** and is illustrated as a bloc scheme in Figure 3.



Source: Landau, I.D., Identificarea și comanda sistemelor, Editura Tehnică, București, 1997

Figure 3. General bloc scheme of a system described through a theta type model

In this model the polynomials *A, B, C, D, F* are defined as such:

$$A(q) = 1 + a_1\cdot q^{-1} + a_2\cdot q^{-2} + \cdots + a_{na}\cdot q^{-na} \tag{1.3}$$

$$B(q) = b_1\cdot q^{-1} + b_2\cdot q^{-2} + \cdots + b_{nb}\cdot q^{-nb} \tag{1.4}$$

$$C(q) = 1 + c_1\cdot q^{-1} + c_2\cdot q^{-2} + \cdots + c_{nc}\cdot q^{-nc} \tag{1.5}$$

$$D(q) = 1 + d_1\cdot q^{-1} + d_2\cdot q^{-2} + \cdots + d_{nd}\cdot q^{-nd} \tag{1.6}$$

$$F(q) = 1 + f_1\cdot q^{-1} + f_2\cdot q^{-2} + \cdots + f_{nf}\cdot q^{-nf} \tag{1.7}$$

the parameter vector being :

$$\theta = \left[a_1, \cdots, a_{na}, b_1, \cdots, b_{nb}, c_1, \cdots, c_{nc}, d_1, \cdots, d_{nd}, f_1, \cdots, f_{nf} \right]^T \tag{1.8}$$

Comparing (1.1) with (1.2), when $k = 0$, we conclude that:

$$\begin{cases} G(q) = \dfrac{B(q)}{A(q) \cdot F(q)} \\ H(q) = \dfrac{C(q)}{A(q) \cdot D(q)} \end{cases} \tag{1.9}$$

The existence of common poles (the zeroes from polynomial *A(q)*) prove that perturbation acts somewhere within the economic process. If degree *na* of polynomial *A(q)* is zero, then the two ways are completely separate and their effect is concentrated on the exit.

Special cases:

1. If *nc=nd=nb=nf=0* then model (1.2) becomes:

$$\begin{cases} A(q) \cdot y(t) = e(t) \\ \theta = \left[a_1, a_2, \cdots, a_{na} \right]^T \end{cases} \tag{1.10}$$

The model given by the relationship (1.10) is named **auto-regressive model (AR).**

2. If *na=nb=nf=nd=0* we obtain a **moving average (MA)** model.

The equation defining this model is:

$$\begin{cases} y(t) = C(q) \cdot e(t) \\ \theta = \left[c_1, \cdots, c_{nc} \right]^T \end{cases} \tag{1.11}$$

3. If *nb=nf=nd=0* then the model is of the **auto-regressive moving average** type **(ARMA)**:

$$\begin{cases} A(q) \cdot y(t) = C(q) \cdot e(t) \\ \theta = \left[a_1, \cdots, a_{na}, c_1, \cdots, c_{nc} \right]^T \end{cases} \tag{1.12}$$

4. If *nf=nc=nd=0* then the model obtained is of the **auto-regressive controlled or with exogenous measures (ARX)**:

$$\begin{cases} A(q) \cdot y(t) = B(q) \cdot u(t-k) + e(t) \\ \theta = \left[a_1, \cdots, a_{na}, b_1, \cdots, b_{nb} \right]^T \end{cases} \tag{1.13}$$

5. If *nd=nf=0* we obtain a **auto-regressive model of moving average with exogenous measures (ARMAX):**

$$\begin{cases} A(q)\cdot y(t) = B(q)\cdot u(t-k) + C(q)\cdot e(t) \\ \theta = \left[a_1,\cdots,a_{na},b_1,\cdots,b_{nb},c_1,\cdots,c_{nc}\right]^T \end{cases} \tag{1.14}$$

The model denominations come from the English **A**uto **R**egressive **M**oving **A**verage with e**X**ogenous control (auto-regressive models with moving average and exogenous control).

6. If *nf=nc=0* we obtain a **ARARX model**. The equation defining this type of model is:

$$\begin{cases} A(q)\cdot y(t) = B(q)\cdot u(t-k) + \dfrac{1}{D(q)}\cdot e(t) \\ \theta = \left[a_1,\cdots,a_{na},b_1,\cdots,b_{nb},d_1,\cdots,d_{nd}\right]^T \end{cases} \tag{1.15}$$

The name ARARX refers to perturbation being modelled as an auto-regressive process, and the dynamics of the system is described by an ARX model.

7. If *na=nc=nd=0* we obtain an **OE model** (**O**utput **E**rror):

$$y(t) = \frac{B(q)}{F(q)}\cdot u(t-k) + e(t) \tag{1.16}$$

8. If *na = 0* we obtain the model **Box – Jenkins**

$$y(t) = \frac{B(q)}{F(q)}\cdot u(t-k) + \frac{C(q)}{D(q)}\cdot e(t) \tag{1.17}$$

The models presented here are mono-variable, specific to SISO (Single Input-Single Output) type economic systems. In the case of multi-variable economic systems (MIMO systems; Multiple Input -Multiple Output), the model (3.2) takes the following equation:

$$A(q)\cdot y(t) = F^{-1}(q)B(q)\cdot u(t-k) + D^{-1}(q)C(q)\cdot e(t) \tag{1.18}$$

where:

$$A(q) = I + A_1\cdot q^{-1} + A_2\cdot q^{-2} + \cdots + A_{na}\cdot q^{-na} \quad \dim A_i = ny\times ny \tag{1.19}$$

$$B(q) = B_1\cdot q^{-1} + B_2\cdot q^{-2} + \cdots + B_{nb}\cdot q^{-nb} \quad \dim B_i = ny\times nu \tag{1.20}$$

$$C(q) = I + C_1\cdot q^{-1} + C_2\cdot q^{-2} + \cdots + C_{nc}\cdot q^{-nc} \quad \dim C_i = ny\times ny \tag{1.21}$$

$$D(q) = I + D_1\cdot q^{-1} + D_2\cdot q^{-2} + \cdots + D_{nd}\cdot q^{-nd} \quad \dim D_i = ny\times ny \tag{1.22}$$

$$F(q) = I + F_1\cdot q^{-1} + F_2\cdot q^{-2} + \cdots + F_{nf}\cdot q^{-nf} \quad \dim F_i = ny\times ny \tag{1.23}$$

having $\dim y = ny \times 1$; $\dim u = nu \times 1$; $\dim e = ny \times 1$, and I being the unit matrix.

Also, unknown parameters are defined by the elements in matrixes $A_i, i = \overline{1,na}$, $B_i, i = \overline{1,nb}$, $C_i, i = \overline{1,nc}$, $D_i, i = \overline{1,nd}$, $F_i, i = \overline{1,nf}$.

The model resulting from equation (1.18) can give way to particular cases perfectly similar to those of the mono-variable system. For this reason, there is no point in repeating the equations of particular multi-variable models.

3. IDENTIFYING ECONOMIC SYSTEMS OFFLINE USING THE LOWEST FOURSQUARE METHOD

From a historical standpoint, the lowest foursquare method was used by Gauss to determine the planets` orbits.

In order to identify an economic system, this method is used to determine the model of the decisive part of a perturbed system taking as criteria the average square modelling error.

To identify the above mentioned method, we will consider that the system to be identified is approximated through a mono-variable ARX model:

$$A(q) \cdot y(t) = B(q) \cdot u(t-k) + e(t) \tag{1.24}$$

where:

$$A(q) = 1 + a_1 \cdot q^{-1} + a_2 \cdot q^{-2} + \cdots + a_{na} \cdot q^{-na} \tag{1.25}$$

$$B(q) = b_1 \cdot q^{-1} + b_2 \cdot q^{-2} + \cdots + b_{nb} \cdot q^{-nb} \tag{1.26}$$

Following, we introduce the notations below:

$$\theta = \left[a_1, \cdots, a_{na}, b_1, \cdots, b_{nb}\right]^T \tag{1.27}$$

$$\varphi(t) = \left[-y(t-1) \ldots - y(t-na) \quad u\left(t-(1+k)\right) \ldots u\left(t-(nb+k)\right)\right]^T \tag{1.28}$$

In these conditions, the equation (1.24) can be written thus:

$$y(t) = \varphi^T(t) \cdot \theta + e(t) \tag{1.29}$$

We can formulate the identification exercise as: knowing the input/output data from the economic system to be identified, please determine the parameters of model ARX so that the following performance index is at a minimum:

$$V(\theta) = \min\left[\sum_{t=1}^{N} e^2(t)\right] = \min\left[\sum_{t=1}^{N}\left(y(t) - \varphi^T(t) \cdot \theta\right)^2\right] \tag{1.30}$$

where N is the available data number.

In view of determining parameters θ, two properties of vector derivative are presented hereunder:

P1. Considering $F(\theta) = c^T \cdot \theta$ where c is a constant vector of dimensions nx1, and θ is the dimension variables vector nx1. In this case we have:

$$\frac{\partial}{\partial \theta}\left[c^T \cdot \theta\right] = c \tag{1.31}$$

P2. Considering $F(\theta) = \theta^T \cdot A \cdot \theta$ where A is a constant and symmetrical matrix of dimensions nxn, and θ is the variable vector of dimensions nx1. In this case we have:

$$\frac{\partial}{\partial \theta}\left[\theta^T \cdot A \cdot \theta\right] = 2 \cdot A \cdot \theta \tag{1.32}$$

As we know**,** the critical points of function (1.30) are determined from the following equation system:

$$\frac{\partial}{\partial \theta}\left[V(\theta)\right] = 0 \tag{1.33}$$

The equation (1.33) can also be written thus:

$$D_1(\theta) = \frac{\partial}{\partial \theta}\left[\sum_{t=1}^{N}\left(y^2(t) - 2 \cdot y(t) \cdot \varphi^T(t) \cdot \theta + \left(\varphi^T(t) \cdot \theta\right)^2\right)\right] \tag{1.34}$$

After proceeding to a derivative in (1.34), we obtain:

$$D_1(\theta) = -2\frac{\partial}{\partial \theta}\sum_{t=1}^{N}\left(y(t) \cdot \varphi^T(t) \cdot \theta\right) + \frac{\partial}{\partial \theta}\sum_{t=1}^{N}\left(\varphi^T(t) \cdot \theta\right)^2 \tag{1.35}$$

The equation (1.35) can also be written thus:

$$D_1(\theta) = D_a(\theta) + D_b(\theta) \tag{1.36}$$

where:

$$D_a(\theta) = -2 \cdot \frac{\partial}{\partial \theta}\left\{\left[\sum_{t=1}^{N}\left(\varphi(t) \cdot y(t)\right)\right]^T \cdot \theta\right\} \tag{1.37}$$

$$D_b(\theta) = \frac{\partial}{\partial\theta}\left\{\theta^T \cdot \left[\sum_{t=1}^{N}\left(\varphi(t)\cdot\varphi^T(t)\right)\right]\cdot\theta\right\} \tag{1.38}$$

By applying the above mentioned derivative properties, the equations (1.37) and (1.38) become:

$$D_a(\theta) = -2\cdot\sum_{t=1}^{N}\left(\varphi(t)\cdot y(t)\right) \tag{1.39}$$

$$D_b(\theta) = 2\cdot\left[\sum_{t=1}^{N}\left(\varphi(t)\cdot\varphi^T(t)\right)\right]\cdot\theta \tag{1.40}$$

In these conditions, the system (1.33) becomes:

$$\left[\sum_{t=1}^{N}\left(\varphi(t)\cdot\varphi^T(t)\right)\right]\cdot\theta = \sum_{t=1}^{N}\left(\varphi(t)\cdot y(t)\right) \tag{1.41}$$

The critical points are gives by the following equation:

$$\hat{\theta} = \left[\sum_{t=1}^{N}\left(\varphi(t)\cdot\varphi^T(t)\right)\right]^{-1}\cdot\left[\sum_{t=1}^{N}\left(\varphi(t)\cdot y(t)\right)\right] \tag{1.42}$$

In order to verify the nature of the critical points (1.42), we will calculate the second order derivative of function $V(\theta)$.

$$\frac{\partial^2}{\partial\theta^2}\left[V(\theta)\right] = \frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}V(\theta)\right]^T \tag{1.43}$$

After calculations we obtain:

$$\frac{\partial^2}{\partial\theta^2}\left[V(\theta)\right] = 2\sum_{t=1}^{N}\left(\varphi^T(t)\cdot\varphi(t)\right) \tag{1.44}$$

From (1.44) we can see that the second order derivative of function $V(\theta)$, is defined positively, indicating the fact that the critical points (1.42) are minimum local points of function $V(\theta)$.

In case the economic system is a multi-variable ARX one, we have:

$$A(q)\cdot y(t) = B(q)\cdot u(t-k) + e(t) \tag{1.45}$$

where:

$$A(q) = I + A_1 \cdot q^{-1} + A_2 \cdot q^{-2} + \cdots + A_{na} \cdot q^{-na} \quad \dim A_i = ny \times ny \tag{1.46}$$

$$B(q) = B_1 \cdot q^{-1} + B_2 \cdot q^{-2} + \cdots + B_{nb} \cdot q^{-nb} \quad \dim B_i = ny \times nu \tag{1.47}$$

We now introduce the following notations:

$$\theta^T = \left[A_1, \cdots, A_{na}, B_1, \cdots, B_{nb}\right] \tag{1.48}$$

$$\varphi^T(t) = \left[-y^T(t-1)\ldots - y^T(t-na) \quad u^T\left(t-(1+k)\right)\ldots u^T\left(t-(nb+k)\right)\right]^T \tag{1.49}$$

In these conditions, the equation (1.45) can also be written thus:

$$y(t) = \theta^T \varphi(t) + e(t) \tag{1.50}$$

Proceeding similarly, the unknown parameters can be determined using the following equation:

$$\hat{\theta} = R^{-1} \cdot \Gamma \tag{1.51}$$

where:

$$\begin{aligned} R &= \sum_{t=1}^{N} \varphi(t)\varphi^T(t) \\ \Gamma &= \sum_{t=1}^{N} \varphi(t) y^T(t) \end{aligned} \tag{1.52}$$

In the equation (1.51), we supposed that the inverse of matrix R exists.

4. OFF-LINE IDENTIFICATION OF PUBLIC EXPENDITURES SYSTEM IN ROMANIA

Romania`s public expenditures system, which will be discussed hereunder is presented in Figure 4.

From Figure 4 we notice that the public expenditures system is a MISO (Multiple Input – Single Output) type one, being identified using the lowest foursquare method applied on a OE (Output Error) model, as shown below:

$$y(t) = F^{-1}(q)B(q) \cdot u(t-nk) + e(t) \tag{1.53}$$

where: $B(q) = B_1 \cdot q^{-1} + B_2 \cdot q^{-2} + \cdots + B_{nb} \cdot q^{-nb}$;

$F(q) = 1 + F_1 \cdot q^{-1} + F_2 \cdot q^{-2} + \cdots + F_{nf} \cdot q^{-nf}$, and nk represents dead time vector expressed in the number of sampling periods.

After identifying the public expenditures system, for vectors *nb*, *nf* and *nk*

nb=[1 3 1 1 0 0 0 0 0 0];
nf=[1 2 2 1 0 0 0 0 0 0];
nk=[1 1 1 1 0 0 0 0 0 0];

we obtain:

$B_1(q) = -0.5505\ q^{-1}$

$B_2(q) = 1.129\ q^{-1} + 6.649\ q^{-2} + 6.711\ q^{-3}$

$B_3(q) = 1.298\ q^{-1}$

$B_4(q) = 3.399\ q^{-1}$

$B_5(q) = 0;\ B_6(q) = 0\ ;\ B_7(q) = 0\ ;\ B_8(q) = 0\ ;\ B_9(q) = 0\ ;\ B_{10}(q) = 0$

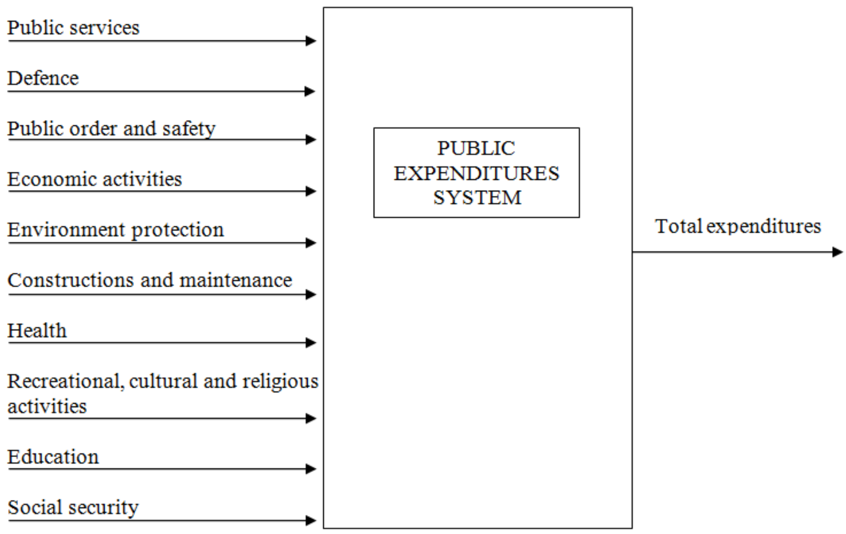
$F_1(q) = 1 - 0.2011\ q^{-1}$

$F_2(q) = 1 + 0.4026\ q^{-1} - 0.5974\ q^{-2}$

$F_3(q) = 1 - 0.3848\ q^{-1} + 0.9936\ q^{-2}$

$F_4(q) = 1 + 0.8224\ q^{-1}$

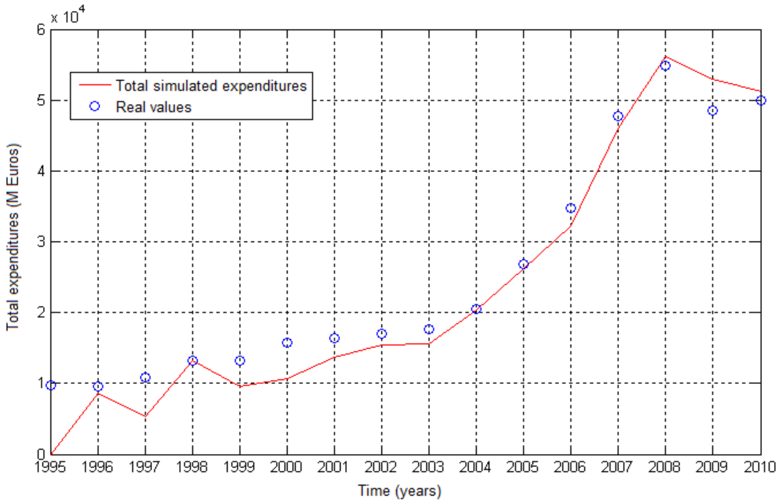
$F_5(q) = 1;\ F_6(q) = 1;\ F_7(q) = 1;\ F_8(q) = 1\ ;\ F_9(q) = 1\ ;\ F_{10}(q) = 1\ .$



Source: Authors

Figure 4. Public expenditures system

When simulating the identified mathematical model (1.53), which has the inputs as in Figure 5, we obtain:

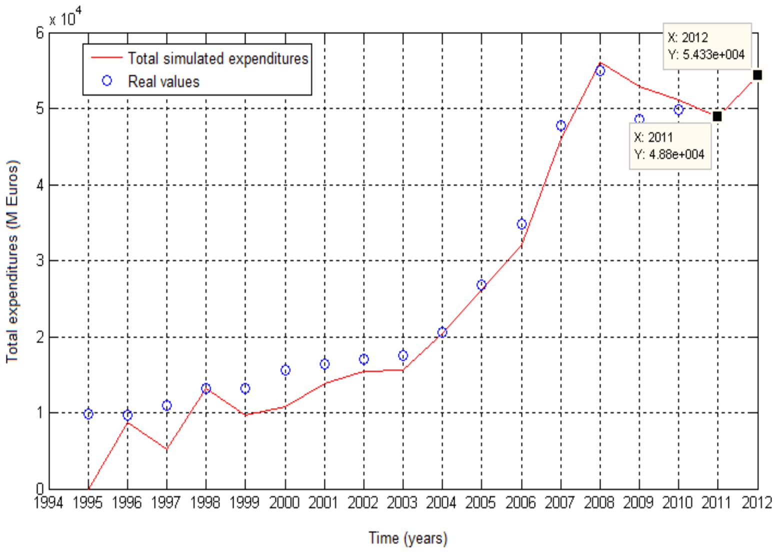


Source: Processing authors based on Eurostat data

Figure 5. Total expenditures evolution in Romania, 1995 – 2010

Based on model (1.53) we can predictions on total expenditures. In this way, if the elements of input vector „u" of the public expenditures system are filled in with components predictions for 2011 and 2012, by simulating the econometrical model above, we can obtain predictions on total public expenditures for the years 2011 and 2012.

After the simulation we have the following chart:



Source: Processing authors based on Eurostat data

Figure 6. Total expenditure predictions for Romania in 2011 and 2012

From Figure 6 we can see that the total expenditure predictions obtain from the simulation are:

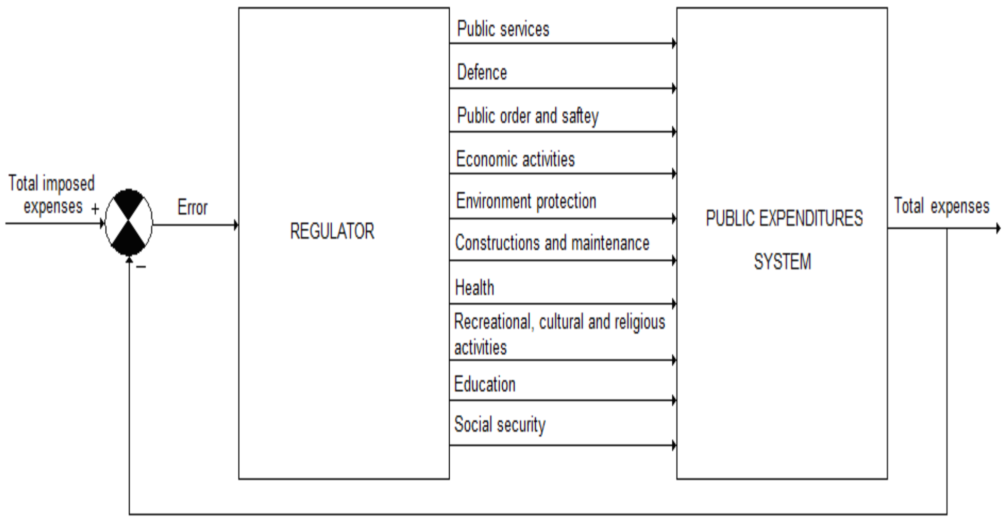
Table 1. Predicted total expenditures

Time [years]	Predicted total expenditures [M Euros]
2011	48800
2012	54330

Source: Processing authors based on Eurostat data

5. OPTIMAL REGULATION OF THE PUBLIC EXPENDITURES SYSTEM IN ROMANIA

Based on the econometric model (1.53), we can design an optimised public expenditures control system. A system as such is presented here below in Figure 7.



Source: Authors

Figure 7. Optimal public expenditures control system in Romania

The central element of the control system presented above is the compensator or regulator. It receives at input the error between imposed public expenditures and real public expenditures (obtained at the output of the public expenditures system).

The regulator processes the error based on certain regulations, also named the laws of regulation, in order to optimise the error between the two expenditures (imposed and real expenditures).

In order to design this regulator, we will determine from (1.53) the mathematical model input – status – output of public expenditures. To determine the mathematical input – status – output model we will use the "Standard Completely Controllable Realization" (SCCR) for MIMO (Multiple Input – Single Output) [6] systems.

Based on method SCCR, we obtain a differential equation system defined by the relation (1.54).

$$\begin{cases} \dfrac{dx(t)}{dt} = A \cdot x(t) + B \cdot u(t) + G \cdot e(t) \\ y(t) = C \cdot x(t) + D \cdot u(t) + e(t) \end{cases} ; \ \mathrm{x}(\mathrm{t}_0) = \mathrm{x}_0 \quad (1.54)$$

where:

- The status measures vector has 7 components and is defined by the following relation:

$$\mathrm{x} = \begin{bmatrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{x}_4 & \mathrm{x}_5 & \mathrm{x}_6 & \mathrm{x}_7 \end{bmatrix}^{\mathrm{T}} \quad (1.55)$$

- The input measures vector has 10 components and is defined by the following relation:

$$\mathrm{u} = \begin{bmatrix} \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 & \mathrm{u}_4 & \mathrm{u}_5 & \mathrm{u}_6 & \mathrm{u}_7 & \mathrm{u}_8 & \mathrm{u}_9 & \mathrm{u}_{10} \end{bmatrix}^{\mathrm{T}} \quad (1.56)$$

where:

- u_1 is the symbol for public services expenditures;
- u_2 is the symbol for defence public expenditures;
- u_3 is the symbol for public order and safety expenditures;
- u_4 is the symbol for economic activities expenditures;
- u_5 is the symbol for environmental protection expenditures;
- u_6 is the symbol for constructions and maintenance expenditures;
- u_7 is the symbol for health system expenditures;
- u_8 is the symbol for amusement, cultural and religious expenditures;
- u_9 is the symbol for education expenditures;
- u_{10} is the symbol for social security expenditures;

The output measures vector "y" has a single component and is defined by the total public expenditures.

By applying the SCCR method for the public expenditures system defined by the relation (1.53) we obtain the following matrixes:

$$A = \begin{bmatrix} -0.6392 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.0870 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.7768 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.2420 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.4729 & 0 & 0 & 0 & 0 & 1 & 0 \\ -0.0981 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.57)$$

$$B = \begin{bmatrix} -0.5505 & 1.1292 & 1.2976 & 3.3987 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4625 & 6.9160 & 1.3286 & -0.6227 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1409 & 8.9493 & -0.6651 & 0.8078 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4560 & 6.2735 & -0.5680 & 1.9760 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0416 & 8.2960 & 0.1282 & -2.4476 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2687 & 3.4777 & 0 & 0.4056 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.1025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.58)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.59)$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.60)$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \quad (1.61)$$

The initial condition of the canonical input – status – output system (1.54) is given by the following vector:

$$x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad (1.62)$$

In order to design the regulator, we will formulate the following exercise:

Problem: Given the linear system (1.54), defined by matrixes (*A,B,C,D,K*), determine a linear system of the following composition:

$$\begin{cases} \dfrac{dx_c(t)}{dt} = A_c \cdot x_c(t) + B_c \cdot E(t) \\ y_c(t) = C_c \cdot x_c(t) + D_c \cdot E(t) \end{cases} \quad (1.63)$$

Named regulator (or compensator), so that the performance index given by the relation (1.64), is at minimum:

$$J = \frac{1}{2} x^T(t_1) \cdot M \cdot x(t_1) + \frac{1}{2} \int_{t_0}^{t_1} \left[x^T(t) \cdot Q \cdot x(t) + u^T(t) \cdot R \cdot u(t) \right] dt \qquad (1.64)$$

where:
Q and M are symmetrical matrixes semi-defined positively, and R is a positive symmetrical matrix.

We can see that the problem enunciated above is a Bolza optimisation type of problem.

In the relation (1.63), the following notations have been used:

E (t) = Total imposed expenditures (t) – Total expenditures (t); y_c (t) = u (t)

(1.65)

The problem above is resolved thus:

1. We estimate the system's status vector (1.54) using the estimate method based on the Kalman filter:

$$\frac{d\hat{x}(t)}{dt} = A \cdot \hat{x}(t) + B \cdot u(t) + K \cdot \left(E(t) - C \cdot \hat{x}(t) \right) \qquad (1.66)$$

where K is the Kalman matrix, and u are Romania's budgetary expenditure system inputs.

2. The Hamiltonian is then formed:

$$H(x,u,\lambda,t) = \frac{1}{2} \cdot \hat{x}^T(t) \cdot Q \cdot \hat{x}(t) + \frac{1}{2} \cdot u^T(t) \cdot R \cdot u(t) + \lambda^T \cdot \left(A \cdot \hat{x}(t) + B \cdot u(t) \right) \qquad (1.67)$$

3. We form the canonical system composed of the equations:

$$\begin{cases} \dfrac{d\hat{x}}{dt} = \dfrac{\partial H}{\partial \lambda} = A\hat{x} + Bu \\ \dfrac{d\lambda}{dt} = -\dfrac{\partial H}{\partial x} = -Q\hat{x} - A^T \lambda \end{cases} \qquad (1.68)$$

4. We extremize the Hamiltonian (1.67) in relation to u:

$$\frac{\partial H}{\partial u} = Ru + B^T \lambda \qquad (1.69)$$

5. from (1.69) results the optimal command:

$$u = -R^{-1} \cdot B^T \cdot \lambda \qquad (1.70)$$

6. In case the extremity t_1 is left alone, the secondary vector λ is chosen thus:

$$\lambda(t) = P \cdot \hat{x}(t) \tag{1.71}$$

where P is the solution to the differentiated Riccati equation:

$$\dot{P}(t) \cdot x(t) + P(t) \cdot \dot{x}(t) = -Q \cdot x(t) - A^T \cdot P \cdot x(t) \tag{1.72}$$

7. Based on the relation (1.71), the expression (1.70) becomes:

$$u(t) = -K_x \cdot \hat{x}(t) \tag{1.73}$$

where: $K_x = R^{-1} \cdot B^T \cdot P$. In these conditions, we obtain:

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ x_i \end{bmatrix} = \begin{bmatrix} A - K_x \cdot B - K \cdot C & -B \cdot K_x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ x_i \end{bmatrix} + \begin{bmatrix} K_x \\ I \end{bmatrix} \cdot E(t) \tag{1.74}$$

$$y_c(t) = \begin{bmatrix} -K_x & I \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ x_i \end{bmatrix} \tag{1.75}$$

where: $x_c = \begin{bmatrix} \hat{x} \\ x_i \end{bmatrix}$; $A_c = \begin{bmatrix} A - K_x \cdot B - K \cdot C & -B \cdot K_x \\ 0 & 0 \end{bmatrix}$; $B_c = \begin{bmatrix} K_x \\ I \end{bmatrix}$;
$C_c = \begin{bmatrix} -K_x & I \end{bmatrix}$; $D_c = 0$

Based on the equations presented above, the regulator used in an optimal budgetary expenditure control is defined by the following matrixes:

$$A_c = \begin{bmatrix} -7.2444 & 0.1060 & -0.2472 & -0.7757 & 0.0929 & -0.1476 & -1.4952 & 1.4356 \\ -12.5412 & -3.2377 & -1.1975 & -2.3094 & -3.4912 & -1.5305 & -13.9216 & 0.6826 \\ -16.8527 & -3.4844 & -4.6887 & -3.2860 & -3.0557 & -1.3440 & -21.6729 & 1.1246 \\ -16.9610 & -2.4580 & -3.6120 & -3.3734 & -0.5426 & -0.2567 & -13.2971 & 1.2752 \\ -10.4611 & -3.4058 & -3.3006 & -2.9395 & -4.1907 & -0.9585 & -19.5444 & 0.0439 \\ -4.7503 & -1.3919 & -1.6452 & -1.5860 & -1.4188 & -1.2805 & -10.1936 & 0.4712 \\ -0.7549 & 0.4498 & 0.5049 & 0.4731 & 0.4526 & 0.2470 & 2.7770 & -0.1130 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(1.66)

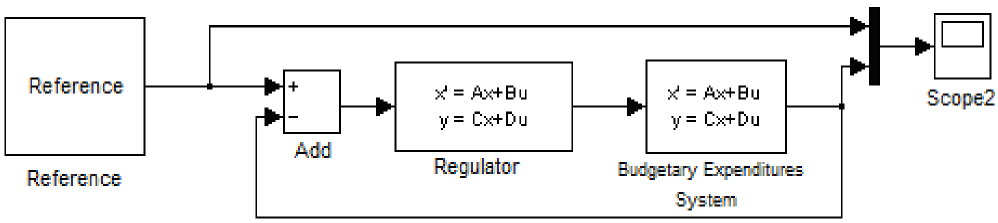
$$C_c = \begin{bmatrix} 0.0359 & 0.1064 & 0.0577 & 0.0028 & -0.4011 & -1.8308 & -7.9830 & -0.0768 \\ -0.0818 & -0.4080 & -0.4580 & -0.4291 & -0.4106 & -0.2240 & -2.5188 & 0.1025 \\ -0.7811 & -0.2781 & 0.6715 & 0.3871 & -0.4953 & -0.6341 & -0.4802 & 0.1101 \\ -0.3811 & -0.0041 & -0.1676 & -0.2330 & 0.2879 & -0.0234 & -0.7128 & 0.3339 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.67}$$

$$B_c = \begin{bmatrix} -4.1842 \\ -11.0709 \\ -15.5499 \\ -16.3639 \\ -11.0891 \\ -4.2226 \\ -0.8452 \\ 1 \end{bmatrix}; \qquad D_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{1.68}$$

These matrixes have been determined knowing that matrixes Q, R and M are:

$$Q = 5 \cdot I_7;\ R = 7;\ M = 0,7 \quad , I \text{ being the unit matrix} \tag{1.69}$$

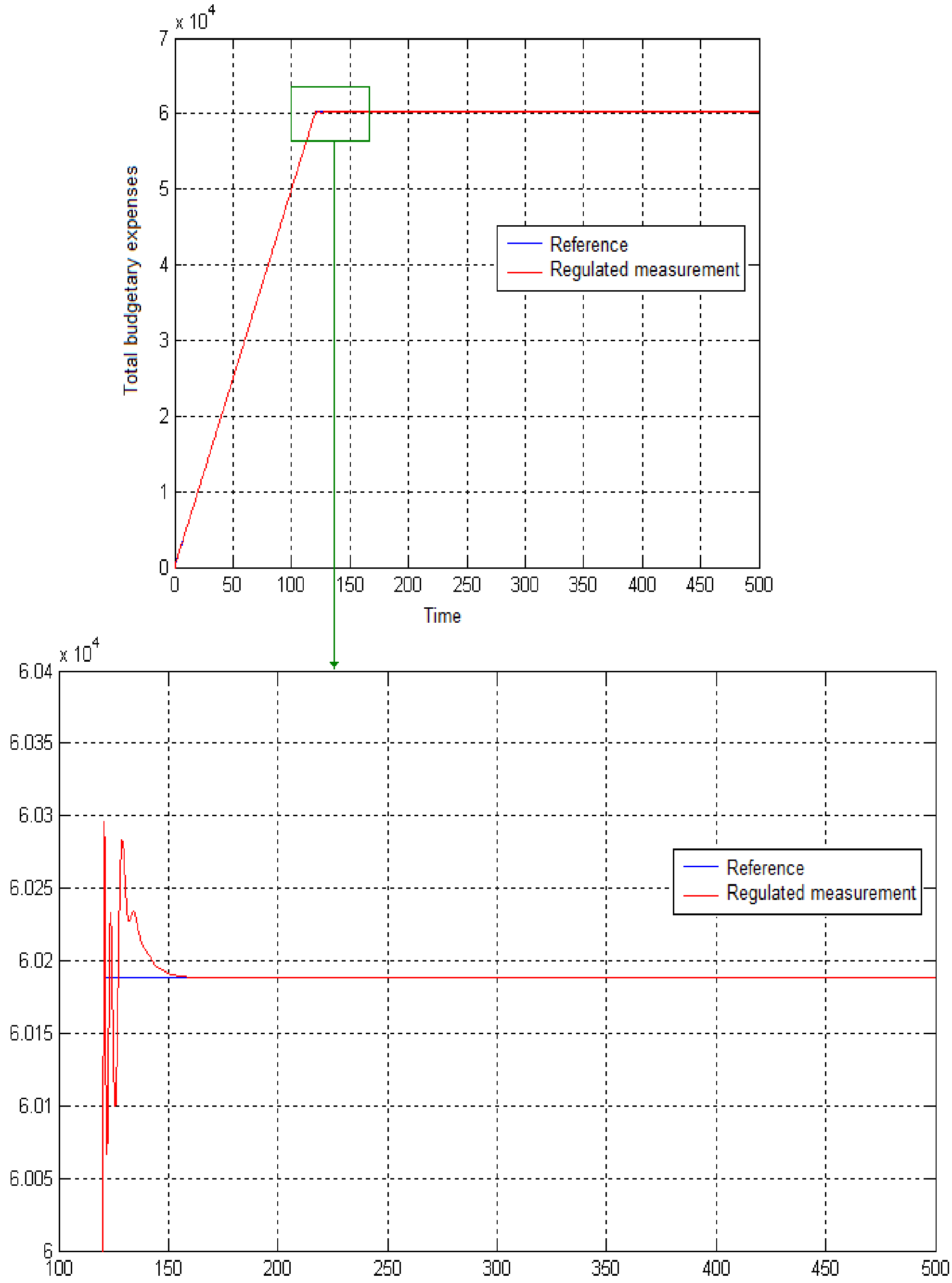
Keeping the above in mind, we can create the simulation for the Matlab Simulink control system based on the following program:



Source: Authors

Figure 8. Simulation program of the public expenditure control system in Romania

After the simulation, we impose a reference measurement for public expenditures of 60188M EURO and we obtain the following result:



Source: Authors

Figure 9. Variation in time of total expenditures in tandem with the reference value

In Figure 9 we have a representation of the variation in time of total public expenditures for a given reference value. We notice that it stabilizes itself after a certain period of time.

6. CONCLUSIONS

In this article we have the description of a public expenditures regulation method in Romania achieved by designing an optimal control system.

The public expenditures control system gives an error that tends to zero, between imposed public expenditures and real expenditures.

Based on this model, for an imposed reference measurement of public expenditures, we can determine the composing elements of the input vector from the mathematical econometric model of Romania's public expenditures system.

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