INTEGER TRIANGLES WITH INTEGER MEDIANS

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***ABSTRACT**: This study introduces the triangles with sides of integer numbers and the results referring to their medians which have as a length integer numbers, too.*

KEYWORDS: *integer triangles, integer medians*

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INTRODUCTION

A notion of geometry in general can be interpreted in many different ways. We think of geometry as of a set of objects and a congruence relation which is normally defined by some group of transformations. For instance, in Euclidean geometry in the plane we study points, lines, segments, polygons, circles, etc, the congruence relation is defined by a group of all length preserving transformations $O(2, \mathbb{R})$ or the orthogonal group.

The similarity between the geometrical figures and arithmetic clearly leads us to taking in consideration those which have the values expressed through integer numbers, or , generally through rational numbers. It is well-known the formula that generates the integer-sided triangles which are not congruent and having the perimeter equal to a given number n. According to Alcuin from York (735-804) $\{t(n)\}$ the Alcuin's sequence that expresses the number of non-congruent triangles which have the perimeter n; for the first values of n we have :

n	0	1	2	3	4	5	6	7	8	9	10
$t(n)$	0	0	0	1	0	1	1	2	1	3	2

It was proved that the function that generates the Alcuin's sequence is :

$$\frac{x^3}{(1-x^2)(1-x^3)(1-x^4)} = \sum_{n=0}^{\infty} t(n)x^n$$

An important role in the developing of the Geometry subject is taken by the computer, many of the results being the consequences of the calculus made with the help of some scientific programs or being part of some scientific projects : for example in project Euler problem 75 https://projecteuler.net/problem=75) which seeks to determinate the numbers of the right triangles with the perimeter equal to n or the problem 257 (http://projecteuler.net/problem=257) which seeks the numbers of the integer triangles ABC with the integer bisectors and the ratio between the area ABC and area AEF rational numbers where E, F are the feet of the bisectors from B and C. The study here tries to present some remarkable results of the geometry of the figures of the integer values.

1.TRIANGLES WITH INTEGER SIDES

Historically speaking, the triangle is one of the mathematical objects which have attracted the attention of the man since antiquity. The collection of 84 problems on the papyrus Rhind (discovered in 1858 in the Valley of the Middle Nile and achieved by the Scottish Egyptologist HenrzRind) dated from 1780-1700 B.C . which contains problems of geometry where the scribe Ahmes presents the calculus formula of the area of an isosceles triangle with the basis of 4 units and the sides of 10 units . In Book 4 from the Arithmetic of Diophantus it appears the problem of finding the right triangle with integer sides and the length of the bisector of one of the sharp angles rational number , having as a solution the triangle with the sides : 7, 24, 25 which cuts the 24 long leg having the length of $\frac{35}{4}$. We should also consider the works of Giovanni Ceva from the 17th century who considers the triangles with integer sides and analyses the Cevians problems (closed intervals which join a vertex with a point from the opposite side) concurrent and rational.

The first mathematician who tries to find integer triangles with integer medians is Euler, the smallest triangles being 2(68,85, 87) and 2 (127, 131, 158)

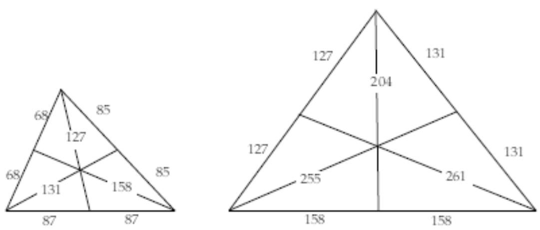


Figure 1.1

It can be noticed that the sides of the big triangle are the double of the medians of the other triangle. In 1779 Euler investigates the triangles with the above-mentioned property, obtaining for its sides the following expressions:

$$a = m(9m^4 + 26m^2n^2 + n^4) - n(9m^4 - 6m^2n^2 + n^4)$$
$$b = m(9m^4 + 26m^2n^2 + n^4) + n(9m^4 - 6m^2n^2 + n^4)$$
$$c = 2m(9m^4 - 10m^2n^2 - 3n^4)$$

In 1813 N. Fuss gives the first **Example** of an integer triangle with the three rational bisectors and, necessarily, the rational area as well $(a = 14, b = 25, c = 25)$ with the bisectors of $i_a = 24$ and

$$i_b = \left(\frac{4ac}{(a+c)^2}p(p-b)\right)^{\frac{1}{2}} = \frac{560}{39} = i_c$$

Definition 1.1. *The triangle ABC is called an integer triangle if its sides have the length integer numbers. The triangle ABC is called rational triangle if its sides have the length rational numbers.*

Theorem 1.1. *For every rational triangle ABC there is a similar integer triangle A' B' C'.*

Proof: If ABC has the sides: $a = \frac{a_1}{a_2},\ b = \frac{b_1}{b_2},\ c = \frac{c_1}{c_2}$, $a_1, a_2, b_1, b_2, c_1, c_2 \in \boldsymbol{Z}$, then the triangle with the sides $a_1b_2c_2,\ a_2b_1c_2,\ a_2b_2c_1$ is similar (we multiplied the sides of the ABC with $a_2b_2c_2$) to ABC.

Theorem 1.2 *In a rational triangle the values of the cosines of the angles are rational numbers.*
Proof: Applying the law of the cosines in ABC with the classical notation we have:

$$a^2 = b^2 + c^2 - 2bc\ cosA$$

where $cosA = \frac{a^2-b^2-c^2}{2bc} \in \boldsymbol{Q}$ and the others as analogue $cosB, cosC \in \boldsymbol{Q}$.

Theorem 1.3 *In a rational triangle the ratio between the sine of two angles is a rational number.*
Proof: From the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin C} = \frac{c}{\sin C} = 2R,$$

We deduce that : $\frac{\sin A}{\sin B} = \frac{a}{b} \in \boldsymbol{Q}$.

Consequence 1.1. *If ABC is a rational triangle which has* $\sin A = r_1\sqrt{n}$ *then*
$\sin B = r_2\sqrt{n},\ \sin C = r_3\sqrt{n}$
where $r_1, r_2, r_3 \in \boldsymbol{Q}$ *and* $n \in \boldsymbol{N}$.
Proof: We have : $\sin B = \frac{b}{a}\sin A = \frac{b}{a}r_1\sqrt{n} = r_2\sqrt{n}$. Analogous for sin C.

Definition 1.2. *Let* $n > 1$ *free of squars. We say that angle A belongs to the class n iff* $sin\ A \in \boldsymbol{Q}[\sqrt{n}]$. *If* $sin\ A \in \boldsymbol{Q}$ *we say that angle A belongs to class 1.*

Observation 1.1. *The rational triangle ABC belongs to class n cu* $n \in \boldsymbol{N}^*, n > 1$ *, n free of squares , if* $sin\ A,\ sinB,\ sinC \in \boldsymbol{Q}[\sqrt{n}]$. *If* $sin\ A,\ sinB,\ sinC \in \boldsymbol{Q}$ *we say that angle A belongs to class 1.*

Example: For $a = 7, b = 5, c = 8$ the triangle belongs to class 3 because $sinA = \frac{1}{2}\sqrt{3}$.
Through direct calculus we find the class of some triangles:

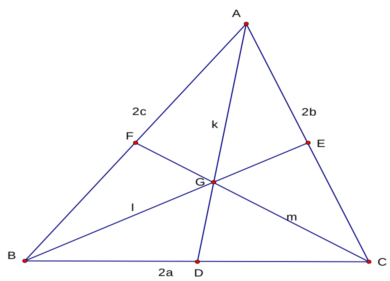
a	*b*	*c*	*Class*
1	*2*	*2*	15
2	*2*	*3*	7
2	*3*	*4*	15
3	*4*	*5*	1
3	*7*	*8*	3
5	*7*	*8*	3
2	*7*	*7*	3
5	*5*	*6*	1

Consequence 1.2. *The rational right triangles belong to class 1.*
Proof: Obviously *sin A=1* , $B = \frac{b}{c} \in \boldsymbol{Q}$.

Theorem 1.4. *If a triangle has the cosines for two angles rational numbers , then the cosine of the third one is rational if and only if the two angles belong to the same class.*
Proof: For *C*, the third angle, we have : $C = 180^0 - A - B$ where $cosC = \sin A\ sinB - cosAcosB$. In conclusion $cosC \in \boldsymbol{Q}$ if and only if $sinAsinB \in \mathbf{Q}$ is equivalent to *A* and *B* from the same class.

2.TRIANGLES WITH INTEGER MEDIANS

In order to find the expressions of the median we will apply the law of the cosine for ABC and ADC with AD as a median :
$AD^2 = AC^2 + DC^2 - 2AC \cdot DC \cdot cos\hat{C}$



$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cdot cos\hat{C}$

We eliminate $cos\hat{C}$ from the two equalities and we obtain the well-known relation of the median:

$4AD^2 = 2AB^2 + 2AC^2 - BC^2$

If the previously mentioned equality stays true for the natural numbers it is necessary that *BC=c* to be even and their analogous as well so that the other medians to be integer. Therefore we say that the triangle ABC has even sides with the lengths: *2a*, *2b*, *2c*.We mark their medians from *A*,*B*, *C* with *k*, *l*, *m*. The relations that give these lengths become :

$$\begin{cases} k^2 = 2b^2 + 2c^2 - a^2 \\ l^2 = 2c^2 + 2a^2 - b^2 \\ m^2 = 2a^2 + 2b^2 - c^2 \end{cases} \qquad (2.1.)$$

By adding them we obtain : $k^2 + l^2 + m^2 = 3(a^2 + b^2 + c^2)$ and the first equation can be written like this : $k^2 = 2b^2 + 2c^2 - a^2 = 2(a^2 + b^2 + c^2) - 3a^2 = \frac{2}{3}(k^2 + l^2 + m^2) - 3a^2$. It follows that the relations that give the expressions of the semi sides according to the medians are :

$$\begin{cases} 9a^2 = 2l^2 + 2m^2 - k^2 \\ 9b^2 = 2m^2 + 2k^2 - l^2 \\ 9c^2 = 2k^2 + l^2 - m^2 \end{cases} \qquad (2.2.)$$

Definition 2.1. *We name a med- triangle an integer triangle with integer medians. We mark the set of the med- triangles with MED.*

Theorem21. *If a, b, c, k, l, m, are natural numbers with* $(a, b, c, k, l, m) = 1$ *then only one of the halves of the sides is an even number.*

Proof: Suppose that two halves of the sides are even and one of them is odd meaning that : $a \equiv 0(mod\ 2)$, $b \equiv 0(mod\ 2)\ and\ c \equiv 1(mod\ 2)$. Then from the equation (2.1.) we get: $k^2 \equiv 2b^2 + 2c^2 - a^2 \equiv 2(mod\ 4)$ which is false because a perfect square is congruent with 0 or 1 modulo 4.

Suppose that all the halves of the sides are odd meaning that $a \equiv 1(mod\ 2),\ b \equiv 1(mod\ 2)\ and\ c \equiv 1(mod\ 2)$. Then the same relation becomes: $k^2 \equiv 2b^2 + 2c^2 - a^2 \equiv 3(mod\ 4)$, again impossible.

It stays true the fact that only one half of the sides is even , the other two being odd.

Theorem 2.2. *The semi perimeter of the triangle MED is even .*

Proof: Pointed in the theorem 2.1.

Theorem 2.3. *Only one median is even the other two being odd.*

Proof: From theorem 2.1 we find out that only one half of the side is even , being marked with a. From the first relation (2.1.) we have k which is even and from the other relations from (2.1.) that l, m are odd.

Theorem 2.4. *Only one of the numbers a, b, c, k, l, m is devisable by 4.*

Proof: According to the previous theorem we can suppose that $a = 2 \propto,\ b = 2\beta + 1,\ c = 2\gamma + 1$ and $k = 2\delta,\ l = 2\lambda + 1,\ m = 2\mu + 1$. Replacing the first relation with (2.1.) we get:

$k^2 + a^2 = 2b^2 + 2c^2$

or

$4\delta^2 + 4\alpha^2 = 8\beta^2 + 8\beta + 2 + 8\gamma^2 + 8\gamma + 2$

from where:

$\delta^2 + \alpha^2 = 2\beta^2 + 2\beta + 2\gamma^2 + 2\gamma + 1$

meaning that $\delta^2 + \alpha^2 \equiv 1(mod\ 2)$ which demonstrates that or $\propto$ or δ are even and or a or k are divisible by 4.

Theorem 2.5. ***(Euler)*** *If the triangle ABC with the sides (2a, 2b, 2c) is in MED and has the medians (k, l, m) then the triangle A'B'C' with the sides (2k, 2l, 2m) is in MED.*

Proof: By marking the medians of the triangle A'B'C' with k' , l', we have:

$$\begin{cases} k'^2 = 2(2c^2 + 2a^2 - b^2) + 2(2a^2 + 2b^2 - c^2) - \\ \qquad (2b^2 + 2c^2 - a^2) = 9a^2 \\ l'^2 = 2(2a^2 + 2b^2 - c^2) + 2(2b^2 + 2c^2 - a^2) - \\ \qquad (2c^2 + 2a^2 - b^2) = 9b^2 \\ m'^2 = 2(2b^2 + 2c^2 - a^2) + 2(2c^2 + 2a^2 - b^2) - \\ \qquad (2a^2 + 2b^2 - c^2) = 9c^2 \end{cases}$$

meaning that the medians of the new triangle are exact *(3a,3b,3c)*. The process can be clearly continued , meaning that if $(2a, 2b, 2c) \in MED$ with the medians *(k, l, m)* then $(2k, 2l, 2m) \in MED$ with the medians *(3a,3b,3c)* and therefore $(6a, 6b, 6c) \in MED$ is a triangle similar to the initial one.

Theorem 3.6. *There are no isosceles triangles in MED.*

Proof: If *a=b* then *k=l* and the equations (2.1.) become:

$$\begin{cases} k^2 = a^2 + 2c^2 \\ m^2 = 4a^2 - c^2 \end{cases} \qquad (2.3.)$$

The second equation tells us that *m* and *c* have the same parity and if they are odd $m^2 \equiv 1(mod\ 4)$ and $4a^2 - c^2 \equiv 3(mod\ 4)$ the equality is impossible . Hence both *m* and *c* are even or $c = 2C,\ m = 2M$ the relations (3.3.) being:

$$\begin{cases} k^2 = a^2 + 8C^2 \\ M^2 = a^2 - C^2 \end{cases}$$

Or replacing the first relation with their difference we get:

$$\begin{cases} k^2 = M^2 + 9C^2 \\ a^2 = M^2 + C^2 \end{cases} \qquad (2.4.)$$

This way we got to the well-known Euler's problem referring to concordant numbers which is stated like this :
The numbers a and b are told to be concordant if there exist integer numbers x, y, z, t with $xy \neq 0$ so that:
$\begin{cases} x^2 + ay^2 = z^2 \\ x^2 + by^2 = t^2 \end{cases}$.
On the contrary (if the numbers x, y, z, t under the above conditions don't exist) the numbers are called DISCONCORDANT.
The problem we got to shows again the fact that the numbers 1 and 9 are or aren't concordant. It is demonstrated [2] that the numbers 1 and 9 are DISCORCONDANT therefore there aren't integer numbers *k, a, M, C* to respect the equations (3.4.).The consequence is that there are no isosceles triangles in the set MED.

Theorem 2.7. *If the triangle ABC is right with* $b^2 + c^2 = a^2$ *then in the triangle with the sides (2a,2b, 2c) we only have a median with the length a natural number.*
Proof : If $b^2 + c^2 = a^2$ the equations (2.1) become:

$$\begin{cases} k^2 = b^2 + c^2 \\ l^2 = 4c^2 + b^2 \\ m^2 = 4b^2 + c^2 \end{cases}$$

where the median from A is obviously *k=a*. Considering the pythagoric numbers a, b, c given by:

$$\begin{cases} a = u^2 + v^2 \\ b = 2uv \\ c = u^2 - v^2 \end{cases}$$

We get for *l* and *m* the following expressions :

$$\begin{cases} l^2 = 4u^4 - 8u^2v^2 + 4v^2 \\ m^2 = u^4 + 14u^2v^2 + v^2 \end{cases}$$

Mordell ([7] pages 20-21) show that the only solutions are for the pairs $(u^2, v^2) \in \{(1,0), (1,1), (0,1)\}$, cases in which the triangle becomes confluent.

Definition 2.2. *We say that the triangle ABC is an automedian triangle if its medians are proportional with its sides.*

Theorem 2.8. *The triangle with the sides 2a, 2b, 2c is automedian if and only if one of the relations :*

$$a^2 + b^2 = 2c^2 b^2 + c^2 = 2a^2 a^2 + c^2 = 2b^2$$

is true.

Proof: Supposing we have the relation : $a^2 + c^2 = 2b^2$ which, replaced in (2.1), gives us:

$$\begin{cases} k^2 = 3c^2 \\ l^2 = 3b^2 \\ m^2 = 3a^2 \end{cases}$$

From here it results that the medians are proportional with its sides.
Vice versa if the medians are proportional with the sides we have :

$$\frac{k}{c} = \frac{l}{b} = \frac{m}{a} = x$$

Then the equations (2.1.) become:

$$\begin{cases} x^2c^2 = 2b^2 + 2c^2 - a^2 \\ x^2b^2 = 2c^2 + 2a^2 - b^2 \\ x^2a^2 = 2a^2 + 2b^2 - c^2 \end{cases}$$

We eliminate X from the first two equations and we get:

$$c^2(2c^2 + 2a^2 - b^2) = b^2(2b^2 + 2c^2 - a^2)$$

Through calculus we get : $(2c^2 + b^2)(c^2 + a^2 - 2b^2) = 0$ and we also get the condition $c^2 + a^2 - 2b^2 = 0$. If we had started with $\frac{k}{a} = \frac{l}{c} = \frac{m}{b} = x$ we would have got $c^2 + b^2 - 2a^2 = 0$ and if we had started with $\frac{k}{b} = \frac{l}{a} = \frac{m}{c} = x$ then we would have found out that : $b^2 + a^2 - 2c^2 = 0$.

Theorem 2.9. *(Euler 1779)The triangle with the sides: (2a, 2b, 2c) gives us the expressions:*

$$\begin{cases} a = (m+n)p - (m-n)q \\ b = (m-n)p + (m+n)q \\ c = 2mp - 2nq \end{cases}$$

where
$p = (m^2+n^2)(9m^2-n^2), \qquad q = 2mn(9m^2 + n^2)$
with $m, n \in \boldsymbol{Z}$ has integer medians.

Observation: Not all the triangles from MED can be obtained through Euler parameter. For example for a=226, b=486, c=580 we have k=523, l=367 m=244 a triangle which does not come from Euler's parameter.

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